Multistage switching fabrics

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Notes for the class on "Switching Technologies for Data Centers"

Politecnico di Torino

September 2021

Outline

Section 1

Space switching

Introduction

Switching contexts

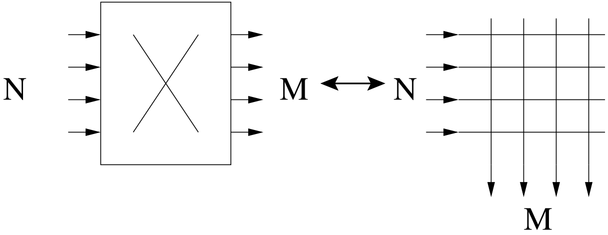
- packet switching (as in the Internet)
- circuit switching (as in the traditional telephone network)

Switching scenarios for different space scaling

- among different processing modules inside a chip
- among chips on the same linecard
- among hosts in a layer-2 network (switch)
- among servers in a data center
- among networks in a layer-3 network (router)

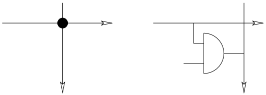
Reference architecture for space switching

- crossbar $N \times M$
  - each internal port may switch an aggregation of external ports (line-grouping)
- best performance
- simple control
- high implementation complexity



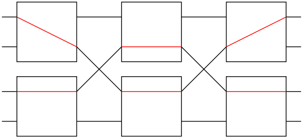
Implementation complexity

- number of basic switching modules
- number of crosspoints
  - related to the number of logical gates and the area on a chip
  - for crossbar: $C(N \times M) = NM$
  - for symmetric crossbar: $C(N) = N^2$
- many other cost functions, depending on the particular technology used for implementation
  - scalability and modularity
  - power consumption
  - reliability
  - switch control and management
  - 2D/3D layout



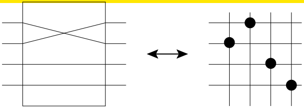
Performance

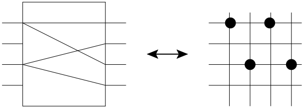
- under *admissible* (i.e., non conflicting) switching requests (circuits or packets)
- non blocking: any input can be always connected to an idle output
    - strictly non blocking (SNB): any new connection does not change the pre-existing connections
    - rearrangeable (REAR): any new connection *may* change some pre-existing connections
- crossbar is SNB by construction
- SNB implies REAR but not viceversa



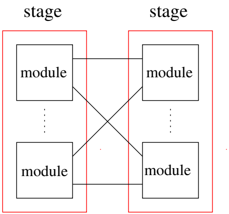
Space switching

- Traffic support
  - Unicast
  - Multicast



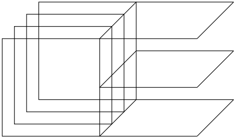


- Multistage networks
  - modules
  - stages

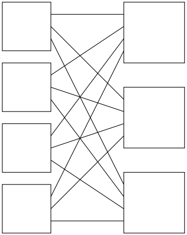


Full interconnections among stages

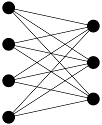
- Two stage switch, with **full interconnections** among the I-stage modules and the II-stage modules
- Possible (equivalent) graphical descriptions:



3D layout



2D layout



graph layout

Section 2

Lee's method (not for 2021-22 academic year)

Lee's method

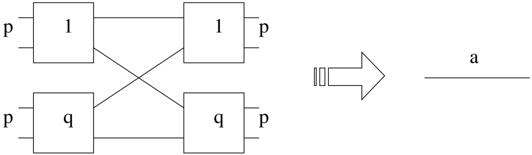
- *approximated* blocking analysis of multistage networks
- assumptions:
  - traffic uniformly distributed among inputs and outputs
  - random routing policy to distribute uniformly the traffic across the modules and links
  - *independence* of the busy state among all the links
- evaluate the blocking probability "seen" by a new circuit to be established, in function of the offered load

Lee's method

- let ρ be the average load of each input (i.e., the fraction of time the input is busy): $\rho \in [0, 1]$
- let $\rho_{tot} = N\rho$ be the total load to the switch: $\rho_{tot} \in [0, N]$ Erlang
- examples
  - in a 10×10 telephone switch, each input receives 6 calls/hour and each call lasts on average 3 minutes; then $\rho = 0.3$ and the total load is $\rho_{tot} = 3$ Erlang
  - in a 10×10 packet switch, with ports at 100 Mbps, each input receives on average 10^3 pkt/s, each of size 1500 bytes; then $\rho = \frac{1500 \times 8 \times 10^3}{10^8} = 0.12$ and $\rho_{tot} = 1.2$ Erlang

Lee's method for two stages

- symmetric network, with $N = pq$ ports
- ρ is the average input load



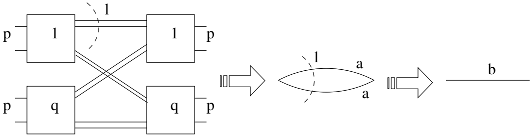
$$a = \frac{\rho N}{q^2} \qquad \Rightarrow \qquad P_b = \rho \frac{p}{q}$$

$$C = 2qN$$

Note that for $\rho \geq \dfrac{q}{p}$, $P_b = 1$.

Lee's method for two stages

- symmetric network, with $N = pq$ ports
- ρ is the average input load
- l multiple edges



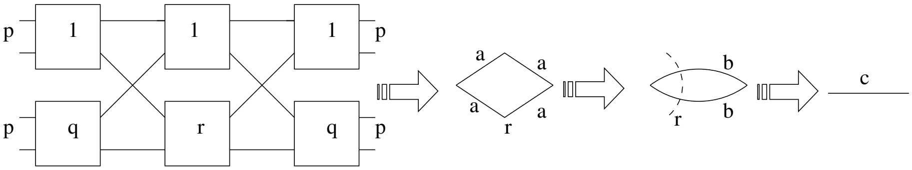
$$a = \frac{\rho N}{lq^2}, \;\; b = a^l \qquad \Rightarrow \qquad P_b = \left(\rho \frac{p}{lq}\right)^l$$

$$C = 2lqN$$

Note that for $\rho \geq \dfrac{lq}{p}$, $P_b = 1$.

Lee's method for three stages

- symmetric network, with $N = pq$ ports
- ρ is the average input load



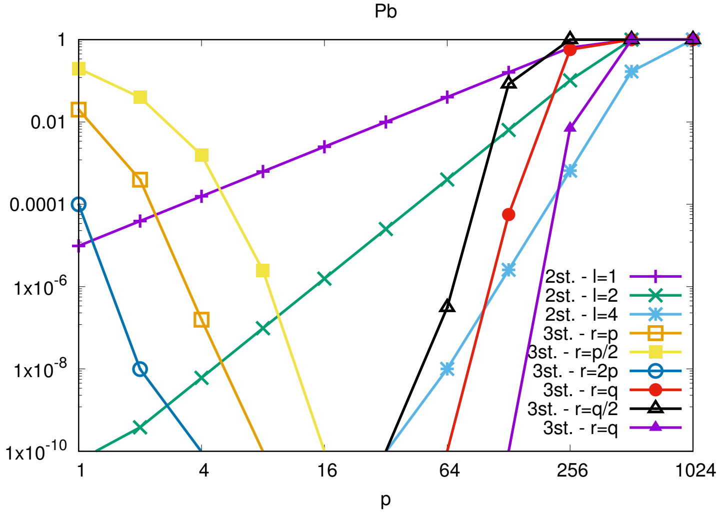
$$a = \frac{\rho N}{qr},\ b = 1 - (1-a)^2 = 2a - a^2,\ c = b^r = a^r(2-a)^r \Rightarrow$$

$$P_b = \rho^r \left[\frac{2N}{qr} - \frac{\rho N^2}{q^2 r^2}\right]^r \qquad C = 2rN + rq^2$$

Note that for $\rho \geq \frac{r}{p}$, $P_b = 1$.

Design comparison

$N = 1024$, $\rho = 0.01$



Design comparison

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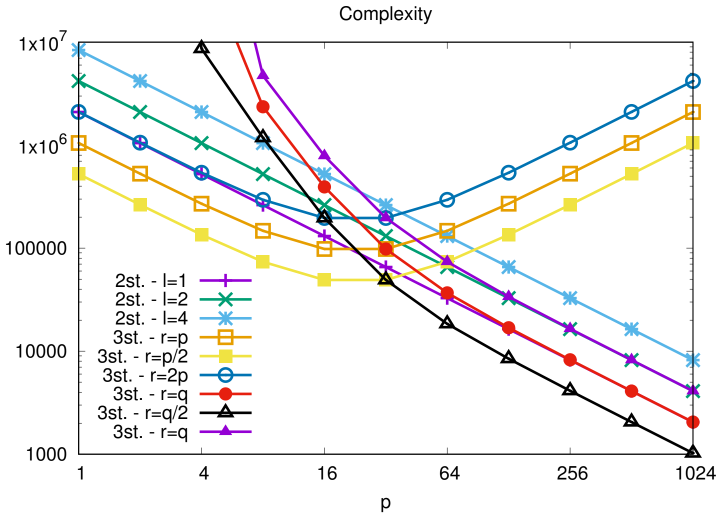


Section 3

Clos networks

Clos networks and derived networks

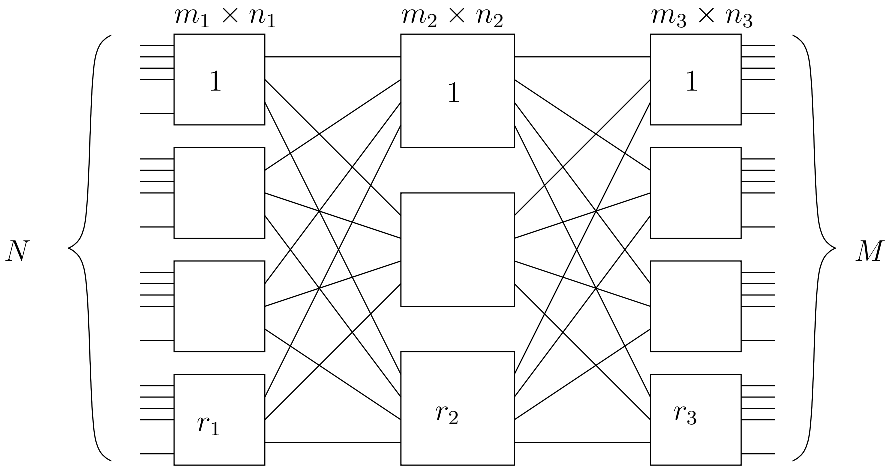
- Clos network
  - strictly non blocking: Clos theorem
  - rearrangeable: Slepian Duguid theorem
  - Paull's matrix and Paull's algorithm
- Recursive construction
  - Benes network ($p = 2$), looping algorithm
  - $p = \sqrt{N}$
- Self routing
  - Banyan networks

Clos networks

- three stage networks
  - m_i: number of inputs for modules at stage i
  - n_i: number of outputs for modules at stage i
  - r_i: number of modules at stage i
  - $M_i = \{1, 2, \ldots, r_i\}$ is the set of modules identifiers belonging to i-th stage
- Exactly one link between two modules in successive stages
  - $r_1 = m_2$, $r_2 = n_1 = m_3$, $r_3 = m_2$

Clos network

$N \times M$ Clos network with $N = m_1 r_1$ and $M = r_3 n_3$



SNB Clos networks

Clos Theorem

A Clos network is SNB if and only if the number of second stage switches r_2 satisfies:

$$r_2 \geq m_1 + n_3 - 1$$

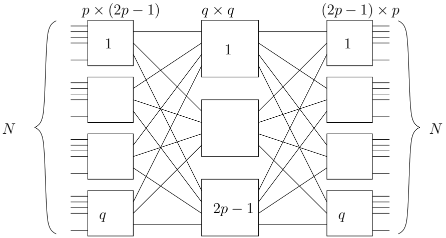
In particular, a symmetric network with $m_1 = n_3 = n$ is SNB if and only if

$$r_2 \geq 2n - 1$$

Proof: Assume that module i of the I-stage should be connected to module j of the III-stage. Hence, a new symbol should be added in P_{ij} of Paull's matrix P. In the worst case, there are already $m_1 - 1$ symbols in the i-th row of P and $n_3 - 1$ symbols in the j-th column. They are all distinct. Hence, to find a new symbol available in the II-stage, it should be $r_2 > (m_1 - 1) + (n_3 - 1)$ which implies $r_2 \geq m_1 + n_3 - 1$.

Complexity of a SNB Clos network

- consider a symmetric Clos network, with $m_1 = n_3 = p$, $r_1 = r_3 = q$ with $N = pq$
- thanks to Clos Theorem, the smallest Clos network is built with $r_2 = 2p - 1$



Total complexity

$$C_{SNB}(N) = qC(p \times (2p-1)) + (2p-1)C(q \times q) + qC((2p-1) \times p) = (2p-1)(2pq + q^2)$$

Approximated complexity (assume $r_2 = 2p$):

$$C_{SNB}(N) \approx qC(p \times (2p)) + 2pC(q \times q) + qC((2p) \times p) = 4p^2q + 2pq^2$$

Paull's matrix

- describes the state of the active interconnections present in a Clos network (i.e., the switching configurations of all the II-stage modules)

Definition

- matrix $P = [P_{ij}]$ of size $r_1 \times r_3$
  - P_{ij} is a set of II-stage modules, i.e. $P_{ij} \subseteq M_2$
  - if $k \in P_{ij}$ means that II-stage module k is connected to I-stage module i and III-stage module j
  - feasibility conditions
    - each row with at most m_1 symbols
    - each column with at most n_3 symbols
    - each element with at most $\min\{m_1, n_3\}$ symbols
    - each $k \in M_2$ appears at most once for each row and for each column

Configuring a SNB Clos network

- when an input of I-stage module i should be connected to an output of III-stage module j, find any II-stage module k such that the connections $i \to k$ and $k \to j$ are both free
  - such connection always exists thanks to the Clos theorem
  - in Paull's matrix P, this operation corresponds to find any available symbol in both i-th row and j-th column

Rearrangeable non-blocking Clos networks

Slepian-Duguid Theorem

A Clos network is rearrangeable (REAR) if and only if the number of second stage switches r_2 satisfies:

$$r_2 \geq \max\{m_1, n_3\}$$

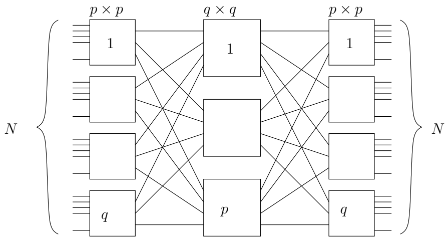
In particular, a symmetric network with $m_1 = n_3 = n$ is rearrangeable (REAR) if and only

$$r_2 \geq n$$

Proof: It will be proved using the Birchkoff von Neumann theorem, later in the course

Complexity of a REAR Clos network

- consider a symmetric Clos network, with $m_1 = n_3 = p$, $r_1 = r_3 = q$ with $N = pq$
- thanks to Slepian Duguid Theorem, the smallest Clos network is built with $r_2 = p$



Total complexity

$$C_{REAR}(N) = qC(p \times p) + pC(q \times q) + qC(p \times p) = 2qp^2 + pq^2$$

Clos complexity comparison

By setting $q = N/p$ in the formulas of the Clos networks complexity:

$$C_{SNB} = (2p-1)\left(2N + \frac{N^2}{p^2}\right) \approx 4pN + \frac{2}{p}N^2$$

$$C_{REARR} = 2pN + \frac{1}{p}N^2$$

and hence,

$$C_{SNB}(N) = \frac{2p-1}{p} C_{REAR}(N)$$

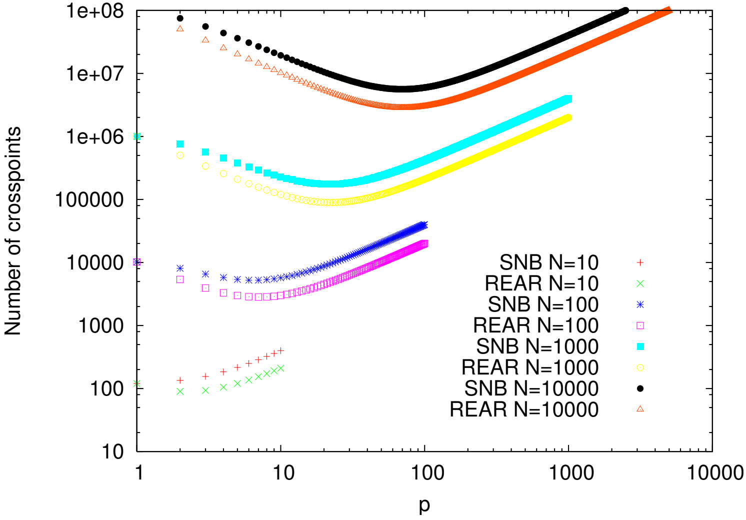
which means:

$$C_{REAR}(N) \leq C_{SNB}(N) < 2C_{REAR}(N)$$

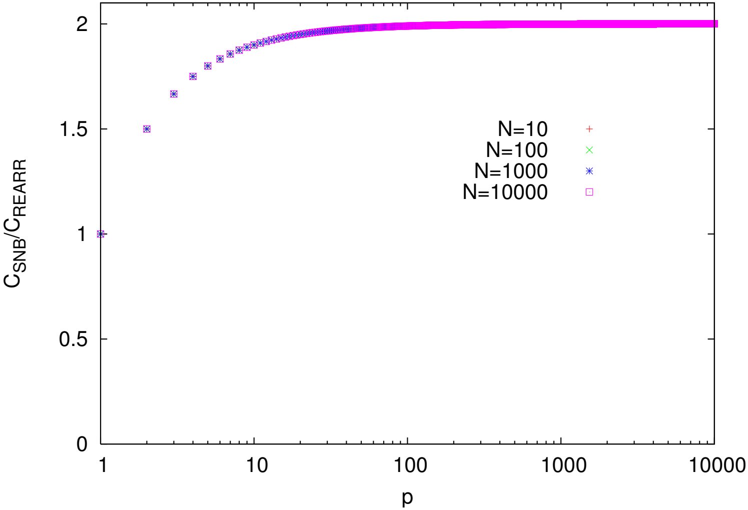
Note that, to be advantageous with respect to the crossbar, it should be:

$$C_{REAR}(N) < N^2 \qquad C_{SNB}(N) < N^2$$

Clos complexity comparison



Clos complexity comparison



Minimum complexity for REARR Clos network

- minimum of C_{REARR} obtained for $\hat{p}$:

$$\frac{\partial C_{\text{REARR}}}{\partial p} = 2N - \frac{N^2}{p^2} = 0 \qquad \Rightarrow \qquad \hat{p} = \sqrt{\frac{N}{2}}$$

  Hence, the minimum complexity is:

$$C_{\text{REARR}}^{opt} = 2\sqrt{2}N\sqrt{N} = \Theta(N\sqrt{N})$$

- for any $N > 8$, $C_{\text{REARR}}^{opt} < C_{\text{crossbar}} = N^2$
- for $p = 1$, the Clos network degenerates into a $N \times N$ crossbar; $C_{REAR}(p = 1) = N^2$
- for $p = N$, the Clos network degenerates into two tandem $N \times N$ crossbars; $C_{REAR}(p = N) = 2N^2$

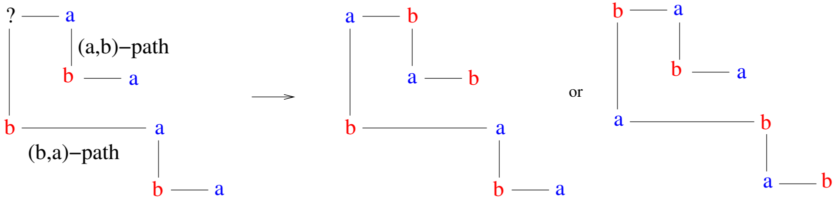
Configuring a REAR Clos network

Paull's algorithm

- incremental algorithm, used to add one connection at one time and reconfigure the network if needed
- will be also used to support rate guarantees in input queued switches
- (**Paull's Theorem**) for each new connection, the number of connections needed to be rearranged is at most $\min\{r_1, r_3\} - 1$
- for each new connection, the number of II-stage modules to reconfigure is at most two

Paull's algorithm

- Given Paull's matrix $P = [P_{ij}]$ and a new connection to add in P_{ij}; two cases are possible
  - it exists a II-stage module a which is available in both row i and column j of P; hence, use module a for the new connection, without any rearrangment: $P_{ij} = P_{ij} \cup a$
  - otherwise, there should be two II-stage modules a and b such that a is available in row i, and b is available in column j of P. Find an (a, b)-path (or a (b, a)-path) starting from P_{ij}. Now swap a with b in such path, and use a (or a b for the (b, a)-path) for the new connection: $P_{ij} = P_{ij} \cup a$.



Section 4

Benes networks

Recursive construction

- main idea to exploit recursively
  - to build a REAR Clos network, use a REAR Clos network for each module
  - to build a SNB Clos network, use a SNB Clos network for each module
- many ways to factorize the network
  - for small complexity, keep small p

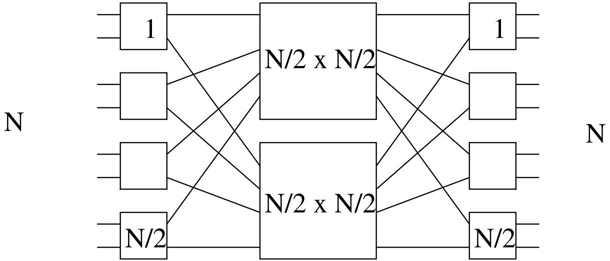
$$C_{REAR}(N) = 2qp^2 + pq^2 \qquad C_{SNB}(N) = (2p-1)q(2p+q)$$

$$C_{REAR}(N, p=2) = N^2/2+4N \qquad \text{vs.} \qquad C_{SNB}(N, p=2) = 3N^2/4+6N$$

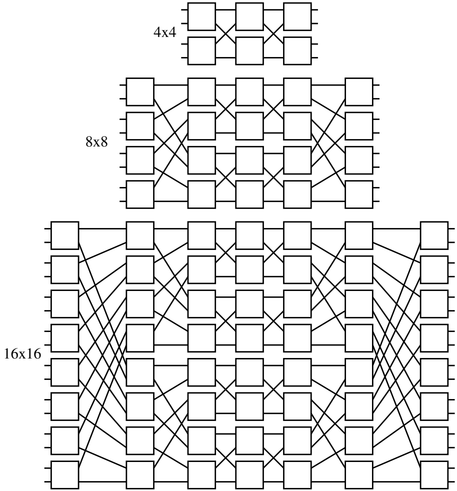
  - for keeping the same "aspect ratio", use $p = \sqrt{N}$

Benes network

- Clos network, REAR, recursively factorized with $p = 2$, exploiting only 2×2 modules
- $N = 2^n$ for some n



Example of Benes networks



Benes network complexity

The number of crosspoints satisfies:

$$C(N) = NC_2 + 2C\left(\frac{N}{2}\right) = kNC_2 + 2^k C\left(\frac{N}{2^k}\right) \qquad \text{for } k = 0, \ldots, \log_2 N - 1$$

Now, by setting $k = \log_2 N - 1$ and considering $C_2 = 4$:

$$C(N) = N(\log_2 N - 1)C_2 + \frac{N}{2}C_2 = 4N\log_2 N - 2N$$

The number of stages satisfies:

$$S(N) = 2 + S\left(\frac{N}{2}\right) = 2k + S\left(\frac{N}{2^k}\right) \qquad \text{for } k = 0, \ldots, \log_2 N - 1$$

and again, by setting $k = \log_2 N - 1$:

$$S(N) = 2\log_2 N - 1$$

Benes network configuration

Two algorithms:

- Paull's algorithm applied recursively
- Looping algorithm
  - equivalent to Paull's algorithm using a particular sequence of switching requests
  - all the switching requests should be known in advance to avoid reconfigurations

Master method for recurrence equations

- Landau notation for $f(n), g(n) > 0$
  - $f(n) = \Theta(g(n))$ means that $\exists\ c, c' > 0, n_0$ s.t. $\forall n \geq n_0$: $cg(n) \leq f(n) \leq c' g(n)$
  - $f(n) = O(g(n))$ means that $\exists\ c > 0, n_0$ s.t. $\forall n \geq n_0$: $f(n) \leq cg(n)$
  - $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$
- Master method to solve $T(n) = aT(n/b) + f(n)$, $a \geq 1$, $b \geq 1$
  - if $\exists\ \epsilon > 0$ s.t. $f(n) = O(n^{\log_b a - \epsilon})$, then

$$T(n) = \Theta(n^{\log_b a})$$

  - if $f(n) = \Theta(n^{\log_b a})$, then

$$T(n) = \Theta(n^{\log_b a} \log_2 n)$$

Clos networks, factorized recursively with factor 2

- REAR Clos network, factorized recursively, $p = 2$ (i.e., Benes network)
  - $C(N) = NC_2 + 2C(N/2)$
  - using master method, $a = b = 2$, then $f(n) = \Theta(N)$; hence,

$$C(N) = \Theta(N \log_2 N)$$

- SNB Clos network, factorized recursively, $p = 2$
  - $C(N) = 2NC_2 + 3C(N/2)$
  - using master method, $a = 3$, $b = 2$, then $f(n) = \Theta(N) = O(n^{\log_2 3 - \epsilon})$ with $\epsilon = 0.5$; hence,

$$C(N) = \Theta(N^{\log_2 3}) \approx \Theta(N^{1.58})$$

REAR Clos network, factorized recursively with factor $\sqrt{N}$

For convenient factorization, assume $N = 2^n$ and $n = 2^k$.

$$C(N) = 3\sqrt{N}C(\sqrt{N}) = 3\ 2^{n/2}C\left(2^{n/2}\right) = 3^k 2^{n/2+n/2^2+\ldots+n/2^k} C\left(2^{n/2^k}\right)$$

If we set $k = \log_2 n$, since $1/2 + 1/2^2 + \ldots + 1/2^k \approx 1$ for large k (i.e., large N)

$$C(N) \approx 3^{\log_2 n} 2^n C(2) = n^{\log_2 3} N C(2) = 4N(\log_2 N)^{1.58}$$

SNB Clos network, factorized recursively with factor $\sqrt{N}$

For convenient factorization, assume $N = 2^n$ and $n = 2^k$. For a better layout, we assume that $r_2 = 2\sqrt{N}$ and then:

$$C(N) = \sqrt{N}C(\sqrt{N} \times 2\sqrt{N}) + 2\sqrt{N}C(\sqrt{N}) + \sqrt{N}C(2\sqrt{N} \times \sqrt{N})$$

Since $C(\sqrt{N} \times 2\sqrt{N}) = 2C(\sqrt{N})$,

$$C(N) = 6\sqrt{N}C(\sqrt{N}) = 6\ 2^{n/2}C\left(2^{n/2}\right) = 6^k 2^{n/2+n/2^2+\ldots+n/2^k} C\left(2^{n/2^k}\right)$$

If we set $k = \log_2 n$, since $1/2 + 1/2^2 + \ldots + 1/2^k \approx 1$ for large k (i.e., large N)

$$C(N) \approx 6^{\log_2 n} 2^n C(2) = n^{\log_2 6} N C(2) = 4N(\log_2 N)^{2.58}$$

Recursive factorization - summary

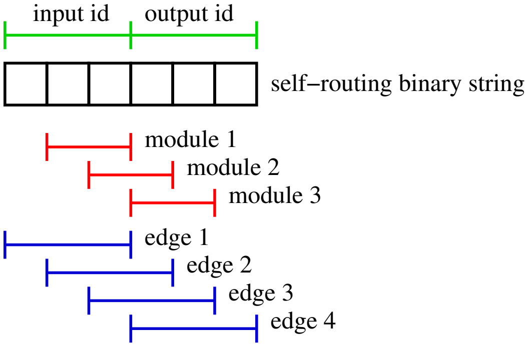
- $p = 2$
  - REAR $\Rightarrow C(N) = 4N \log_2 N$ (Benes)
  - SNB $\Rightarrow C(N) = \Theta(N^{1.58})$
- $p = \sqrt{N}$
  - REAR $\Rightarrow C(N) = 4N(\log_2 N)^{1.58}$
  - SNB $\Rightarrow C(N) = 4N(\log_2 N)^{2.58}$

Section 5

Self-routing networks

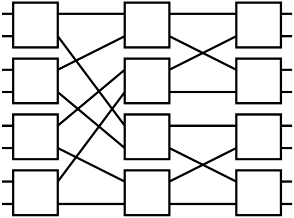
Banyan networks

- self-routing $N \times N$ switch
  - header of the packet drives the routing path
- complexity $\Theta(N \log_2 N)$
- unique path from each input to each output
- based on the Benes network

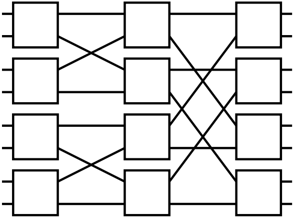


Examples of Banyan networks

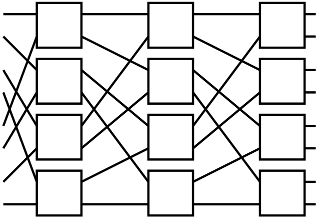
Baseline network



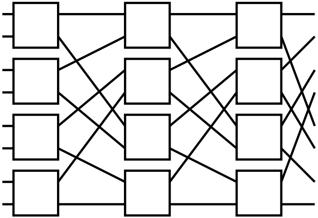
Banyan network



Shuffle exchange (Omega) network



Flip (inverse shuffle exchange) network



Blocking in Banyan networks

- **Property:** if self-routing addresses satisfy both conditions:
  - strictly monotone outputs, i.e. output destinations are increasing at the inputs
  - compact monotone inputs, i.e. no idle inputs between any two active inputs

  then the self-routing is non-blocking
- in general, Banyan networks are blocking
- it can be shown that the probability that a random input-output permutation is non-blocking is $2^{-(N/2)\log_2 N}$ which goes to zero very quickly by increasing N
  - i.e., most full switching configurations are blocking

Batcher-Banyan network

Two switching phases:

- (1) self-sorting Batcher network
  - transform any switching request into a non-blocking switching request for Banyan network
- (2) self-routing Banyan network
- final complexity $= \Theta(N(\log_2 N)^2)$