Applications to connectivity and capacity of wireless multi-hop networks

Olivier Dousse, Matthias Grossglauser and Patrick Thiran

CH-1015 Ecublens
Patrick.Thiran@epfl.ch
http://icawww.epfl.ch

Content

- Only static networks
- Only connectivity and throughput (capacity)
- Full vs partial connectivity

1. From Boolean model to STIRG (Signal To Interference Ratio Graph)

2. Transport capacity a la Gupta-Kumar (interference = foe)
   - Upper bound
   - Lower bound: full connectivity vs percolation

3. Relay channel (interference = friend)
The simplest model: Boolean model

- Position of nodes is a (often homogeneous Poisson) spatial point process, with intensity $\lambda$.
- Boolean model: fixed radio range $r$. Nodes $i$ and $j$, at positions $x_i$ and $x_j$ are directly connected iff

$$\|x_i - x_j\| < r$$

Full vs partial connectivity

- Long range connectivity appears much before full connectivity because of percolation. Probability of connectivity $P_c \approx \Theta^2$ for nodes located far away from each other ($P_c \Theta^2$ by FKG), where $\Theta$ is the percolation probability.
Beyond the Boolean model

- What actually matters is not the distance between nodes, but the rate $R$ [bits/sec] at which they can exchange data.
- When 1 source and 1 destination, Shannon’s theorem states that the maximal rate $R = 1/2 \log(1 + \text{SNR})$ is achievable, where $\text{SNR} = \frac{\text{PL}(d)}{N_0}$ is the Signal to Noise Ratio of the channel:
  - $P$ = Emitting power of source.
  - $L(d) = \text{Attenuation function at distance } d \text{ between the nodes (e.g., } L(d) = d^{-\alpha})$
  - $N_0 =$ Background thermal noise
- When there are more than 2 nodes, the problem has been solved only in very particular cases.

Two models for wireless multi-hop networks

- Multiuser information theory is confronted with the problem of interferences. Friend or foe?
- Approach 1 (Physical model from Gupta-Kumar)
  Interferences = foe. A link exists between Nodes i and j ifff the Signal to (Noise + Interference) Ratio is larger than some hardware dependent threshold. Enables
  - Computation of the connectivity graph
  - Lower bounds on the achievable capacity
- Approach 2 (Relay channel) Interferences = friend. A link exists between Nodes i and j if they can communicate at rate $R$, possibly using all other nodes as relay. Enables
  - Computation of the connectivity graph
  - Pure information-theoretic upper bounds on capacity
Signal To Interference Ratio physical model (interference = foe)

- Signal to Noise Ratio at Node j receiving from Node i is

\[ \text{SNR}_{i\rightarrow j} = \frac{P_i L(\|x_i - x_j\|)}{N_0 + \gamma \sum_{k \neq i, j} P_i L(\|x_k - x_j\|)} \]

- \( P_i \) = Emitting power of node i. Here \( P_i = P \) (no power control)
- \( L(d) \) = Attenuation function at distance d (e.g., \( L(d) = d^{-\alpha} \))
- \( N_0 \) = Background thermal noise
- \( \gamma \) = degree of orthogonality of the code (\( \gamma = 1 \) for a narrowband system, \( 0 \leq \gamma < 1 \) for a CDMA system)

- Nodes i and j are directly connected iff

\[ \min\{\text{SNR}_{i\rightarrow j}, \text{SNR}_{j\rightarrow i}\} > \beta \]

A more general model than the Boolean model

- Signal to Noise Ratio at Node j receiving from Node i is

\[ \text{SNR}_{i\rightarrow j} = \frac{P_i L(\|x_i - x_j\|)}{N_0 + \gamma \sum_{k \neq i, j} P_i L(\|x_k - x_j\|)} \]

- \( \gamma = 0 \). Then Nodes i and j are directly connected iff

\[ \frac{P_i L(\|x_i - x_j\|)}{N_0} > \beta \Leftrightarrow \|x_i - x_j\| < r_b = L^{-1}(\beta N_0 / P) \]

Identical to the Boolean model.
Interference = shot noise

- Signal to Noise Ratio at Node j receiving from Node i is

\[
\text{SNR}_{i \rightarrow j} = \frac{P_i L(\|x_i - x_j\|)}{N_0 + \gamma \sum_{x_k} P_k L(\|x_k - x_j\|)}
\]

- The interference term is a shot noise
- \(\gamma > 0\). If \(L(x)\) is such that

\[
\int_{\gamma}^{\infty} xL(x)dx = \infty
\]

for any \(\gamma > 0\) (e.g. \(L(d) = d^{-\alpha}\) with \(\alpha \leq 2\)), then the graph is almost surely disconnected (D. Daley, 1971) - Olbers' paradox
- Does percolation still hold if \(\gamma > 0\), but small enough?

Interferences can destroy connectivity...

\(\gamma = 0\)

\(\gamma > 0\)
... but not always.

- Differences with the Boolean model:
  - The existence of an edge depends on every other node's position.
  - The node degree is bounded (it was Poisson for the Boolean model) by $1 + 1/\beta \gamma$. A necessary condition for percolation is thus $\gamma < 1/\beta$.

Main result

- Assumptions:
  - $L(x)$ is a decreasing function of $x$.
  - There is some $0 < d < \infty$ such that $L(x) = 0$ for $x > d$.
  - There is some $0 < \delta < d$ and $M > \beta N_0/P$ such that $\beta N_0/P < L(x) \leq 1$ for $x < \delta$.

- Theorem 1 (Dousse, Baccelli, Thiran (IEEE Trans Networking 2005)):
  There exists $\lambda' < \infty$ and a function $\gamma'(\lambda)$ such that for all $\lambda > \lambda'$:
  - $\gamma'(\lambda) > 0$.
  - For all $\gamma < \gamma'(\lambda)$ the graph contains an infinite cluster a.s.

- Critical threshold $\gamma'(\lambda) \geq \gamma'(\lambda)$.
**Mapping on a lattice**

- Divide the plane in squares of size $d/2 \times d/2$:
  - $LR = \{LR \text{ crossing of rectangle } 3d/2 \times d/2\}$
  - $TB_1 = \{TB \text{ crossing of left square } d/2 \times d/2\}$
  - $TB_2 = \{TB \text{ crossing of right square } d/2 \times d/2\}$
  - Open edge probability $p = P(LR \cap TB_1 \cap TB_2)$

- If $\pi r^2 \lambda > (\pi r^2 \lambda)_c$, then by taking $d$ sufficiently large, $p$ can be made as large as we want.

- Edges are not independent, but only adjacent edges are dependent (1-dependent). (\text{\textit{\textless} dominate an independent bond model, see Liggett}).

- For any integer $m$, for any collection of $n$ edges $a_1, a_2 \ldots a_n$ can pick a subsequence of non adjacent edges $a_{k1}, a_{k2} \ldots a_{km}$ ($m \leq n/4$) that are independent:
  \[
P(\{a_1 \text{ closed}\} \cap \ldots \cap \{a_n \text{ closed}\}) \leq P(\{a_{k1} \text{ closed}\} \cap \ldots \cap \{a_{km} \text{ closed}\}) = (1-p)^m \leq (1-p)^{n/4}
  \]
  can be made as close to 1 as we want, by picking $d$ and thus $p$ sufficiently large.

---

**Mapping on a lattice**

- Distinguish two events for an edge $a$ to be open:
  - $A_a = 1$ if edge $a$ is open in the Boolean model (without interference))
  - $B_a = 1$ if the amount of interference at any point in the rectangle $3d/2 \times d/2$ is less than some value $M$
  - $C_a = A_a B_a$

- Take $r_b = L^{-1}(\pi N_0/P) \pi r^2 \lambda > (\pi r^2 \lambda)_c / \lambda$ then $p_A = P(A_a = 1)$ can be made as large as we want by picking $d$ large enough.

- For any collection of $n$ edges $a_1, a_2 \ldots a_n$, let $q_A = (1-p_A)^{1/4}$, then
  \[
P(A_{a1} \cap \ldots \cap A_{an} = 0) \leq (1-p_A)^n = q_A^n
  \]
  can be made as small as we want by picking $d$ large enough.
Bond percolation - Dual technique

- Construct dual lattice $L_d$ of $L^2$.
- Pick integer $m$, and box $B(m)$:
  - $F_m = \{\text{closed circuit in $L_d$ encircling $B(m)$}; F_m \text{ does not occur}\}$
  - $G_m = \{\text{all edges of $B(m)$ are open}\}$
- $\theta(p) \geq p_p(F_m \cap G_m) = p_p(F_m) p_p(G_m)$
- $p_p(G_m) > 0$ (finite)
- Need to compute $p_p(F_m)$
- Need first to compute the number of closed circuits of length at least $8m$

Bond percolation: Peirls’ argument

- Can construct the circuit a self-avoiding walk of length $n-1$ starting and ending at one of these $n/2$ edges.
- Number of such circuits $\leq (n/2) \cdot \alpha(n-1)$
- $p_p(F_m) \leq \sum_{n=8m}^{\infty} n \cdot p_p(3 \text{ closed circuit of length } n)$
  - $\leq \sum_{n=8m}^{\infty} (n/2) \cdot \alpha(n-1) \cdot p_p(\text{all } n \text{ edges of the circuit are closed})$
  - $\leq \sum_{n=8m}^{\infty} (n/2) \cdot \alpha(n-1) \cdot q_n n$
  - $\leq (2q_n/3) \cdot \sum_{n=8m}^{\infty} n(3q_n)^{n-1}$
  - $\leq 1/2$
- if $m$ is large enough and $q_n < 1/3$.
- $\theta(p) \geq p_p(F_m) p_p(G_m) \geq p_p(G_m)/2 > 0$. 

© Patrick Thiran, LCA, EPFL
Reverse Mapping from the grid

- If the bond model is super-critical, then the Boolean model is also supercritical.
- But interferences were neglected

Interferences

- $B_a = 1$ if the amount of interference at any point in the rectangle $3d/2 \times d/2$ is less than some value $M$.
- Apply Chernoff bound + Campbell Theorem:
  - For any collection of $n$ edges $a_1, a_2, ..., a_n$, we can bound $P(B_{a_1} = 0 \cap ... \cap B_{a_n} = 0) \leq q_B^n$
  - where
    \[
    q_B = \exp\left(\frac{4\pi\lambda}{K} \int_0^\infty L(x)xdx - \frac{M}{K}\right)
    \]
    
    $I = \sum_i PL(x) = 4 + \sum_i (4 + 8i)L\left(\frac{id}{\sqrt{2}}\right) = K$

- $q_B$ can be made as small as we want by picking $M$ large enough, provided for some $y > 0$
  \[
  \int_y^\infty xL(x)dx < \infty
  \]
Mapping on a lattice

- Distinguish two events for an edge $a$ to be open:
  - $A_a = 1$ if edge $a$ is open in the Boolean model (without interference)
  - $B_a = 1$ if the amount of interference at any point in the rectangle $3d^2/2 \times d/2$ is less than some value $M$
  - $C_a = A_a B_a$

- Decouple $A_a$ and $B_a$ using Schwarz's inequality

- For any collection of $n$ edges $a_1, a_2, ..., a_n$, we can bound
  $$P(C_{a_1} = 0 \cap ... \cap C_{a_n} = 0) \leq q_c^n$$
  where
  $$q_c = \sqrt{q_A} + \sqrt{q_B}$$

- $q_c$ can be made as small as we want by picking $d$ and $M$ large enough.

Bond percolation: Peirls' argument

- Can construct the circuit a self-avoiding walk of length $n-1$ starting and ending at one of these $n/2$ edges.
- Number of such circuits $\leq (n/2) \sigma(n-1)$

- $P($closed circuit$) 
  \leq \sum_{n=4}^{\infty} P(\exists$ closed circuit of length $n$) 
  \leq \sum_{n=4}^{\infty} (n/2) \sigma(n-1) P($all $n$ edges of the circuit are closed$) 
  \leq (2q_c/3) \sum_{n=4}^{\infty} n(3q_c)^{n-1} 
  < 1$ 
  if $q_c \leq 2/9$. 
Actually we have shown more

- Assumptions:
  - \( L(x) \) is an isotropic, continuous, decreasing function of \( x \)
  - \( L(x) = 0 \) for \( x > d \)
  - \( L(x) \leq 1 \)

- Theorem 1' (Dousse, Franceschetti, Macris, Meester, Thiran): For all \( \lambda > \lambda_c \) (Boolean threshold), there exists
  - \( \gamma' (\lambda) \geq 0 \)
  - for all \( \gamma < \gamma' (\lambda) \) the graph contains an infinite cluster a.s.

- Critical threshold \( \gamma^*(\lambda) \geq \gamma' (\lambda) \)

What happens at high density?

- For large values of \( \lambda \), interferences dominate.
- Theorem 2: There exist two constants \( c_1 < c_2 \) such that
  \[ \frac{c_1}{\lambda} \leq \gamma^*(\lambda) \leq \frac{c_2}{\lambda} \text{ for } \lambda \to \infty \]
- Lower bound follows from Theorem 1
- Upper bound is obtained by mapping on lattice and using site percolation
Proof of upper bound

- Mapping of plane on a grid
  - Divide the plane in squares of size $\delta/2 \times \delta/2$ ($\beta N_0/P < L(x) < M$ for $x < \delta$).
  - Number of points in a square is $N' \sim \text{Poisson}(\lambda \delta^2/4)$.
- A square is "over-populated" if all the nodes are isolated (holds if $N > (1+2\beta \gamma)PM/\beta \gamma N_0$).
- Define a square to be closed (resp. open) iff it is (resp. not) over-populated.
- For $0 < \varepsilon < 1$, $P(N' \leq (1-\varepsilon)\lambda \delta^2/4) \rightarrow 0$ as $\lambda \rightarrow \infty \Rightarrow P(\text{square open}) \approx 0.59$ for $\lambda$ large enough.
- The origin $O$ is a.s. surrounded by a chain of closed squares and belongs to a finite cluster.

- $O$ belongs to a finite cluster a.s if $(1+2\beta \gamma)PM/\beta \gamma N_0 \leq (1-\varepsilon)\lambda \delta^2/4$ i.e., if $\gamma \leq c_2/\lambda$ with $c_2 = 4PM/\beta \gamma N_0(1-\varepsilon)\delta^2$.
- Geometric arguments: no pair of nodes can be connected across the chain of closed squares. Hence no percolation in STIRG.

What happens at low density?

- For small values of $\lambda$, attenuation (distance between nodes) is more important than interferences.
- $\gamma = 0$ (interferences neglected) $\Rightarrow$ STIRG = Boolean model, with radius $r = L^{-1}(\beta N_0/P)$. The critical density of Boolean model is $\lambda_c$.
- What happens for $\lambda_c < \lambda < \lambda'$?
- Theorem 1 (Dousse, Franceschetti, Macris, Meester, Thiran): For all $\lambda > \lambda_c$ (Boolean threshold), there exists
  - $\gamma' (\lambda) > 0$
  - for all $\gamma < \gamma' (\lambda)$ the graph contains an infinite cluster a.s.
Assumptions can be relaxed. In particular, can take bounded power law attenuation function over unbounded domains (Ongoing work).

Simulation with \( L(x) = \max(1, x^{-3}) \)

Shape of attenuation function \( L(x) \) is important.

Strict power law function \( L(x) = x^{-\alpha} \):
- Multiplying \( \lambda \) by \( a \), or equivalently dividing distance by \( \sqrt{a} \), amounts to replace \( N_0 \) by \( a^{-\alpha} N_0 \rightarrow a \rightarrow \infty, a \rightarrow N_0 \rightarrow 0 : \) connectivity improves !
- \( \gamma^*(\lambda) \) can only be a non decreasing function of \( \lambda \).

Bounded attenuation function, ex: \( L(x) = \min(1, x^{-\alpha}) \):
- Moderate densities: SNIR is larger than with unbounded \( L(x) = x^{-\alpha} \)
- High densities: SNIR goes to zero
Critical curve $\gamma^*(\lambda)$: impact of attn fctn

- Simulation with $L(x) = \min(1, x^{-3})$ and $L(x) = x^{-3}$

TDMA vs CDMA

- The value of $\gamma$ may be required to be very small
- Alternative: TDMA scheme
  - Time intervals divided in $n$ time slots
  - Every node randomly picks a time slot among these $n$ slots to emit
- Assumptions:
  - $L(x)$ is a decreasing function of $x$
  - There is some $0 < \delta < d$ and $M > m > 0$ such that $m < L(x) < M$ for $x < \delta$
- Theorem 4: There exists $\lambda' < \infty$ and a constant $c_3$ such that for all $\lambda > \lambda'$ and $n \geq c_3 \lambda$, the $n$ superposed TDMA graphs contain an infinite cluster a.s.
- TDMA achieves the same connectivity as CDMA with $\gamma/n$
## Content

1. From Boolean model to STIRG (Signal To Interference Ratio Graph)

2. Transport capacity a la Gupta-Kumar (interference = foe)
   - Upper bound
   - Lower bound: full connectivity vs percolation
   - P. Gupta, P.R. Kumar, "Transport Capacity of Wireless Multi-Hop Networks", IEEE Transactions on Information Theory, 2000

3. Relay channel (interference = friend).

---

### Throughput

**Physical model:**
- Signal to Noise Ratio at Node $j$ receiving from Node $i$ is
  \[ SNR_{i\rightarrow j} = \frac{P_i \| x_i - x_j \|}{N_0 + \sum_{k \neq i} P_k \| x_i - x_k \|} \]
  - $P_i$: Emitting power of node $i$
  - $L(d)$: Attenuation function at distance $d$ (e.g., $L(d) = d^{-\alpha}$)
  - $N_0$: Background thermal noise
  - $\gamma = 1$ for a narrowband system
  - Nodes $i$ and $j$ are directly connected iff
    \[ \min \{SNR_{i\rightarrow j}, SNR_{j\rightarrow i} \} > \beta \]

**Connectivity:** fix $\beta$ what $\gamma(\lambda)$ can you afford to have percolation?

**Dual:** fix $\gamma$ what $\beta(\gamma)$ can you afford to have percolation?

**Dual:** fix $\lambda$ and topology: what $\beta(\gamma)$ can you afford to have this topology?

**Related to capacity,** as Shannon capacity of link is $1/2 \log(1 + \beta)$
Throughput

- Connectivity: fix $\beta$ what $\gamma$ can you afford to have percolation?
- Dual: fix $\lambda$ and topology: what $\beta(\gamma)$ can you afford to have this topology?
- Related to capacity, as Shannon capacity of link is $1/2 \log(1 + \beta)$

![Graph showing throughput and interference limited vs. noise limited]

Transport capacity

- $n$ nodes distributed on an area $A$ ($\lambda = n/A$).
- We will take $n \to \infty$. Two possible settings:
  - Dense network: $A = 1$ and $\lambda \to \infty$.
  - Extended network: $A = n$ and $\lambda = 1$ (or $A = n/cst$ and $\lambda = cst$).
- What is the throughput $\Theta(n)$ achievable per node with a uniform traffic matrix?
- Transport capacity: bit x meter /sec
  - Dense network: distances between end nodes are $O(1)$ so that $\Theta(n)$ is indifferently in bit/sec or bit x meter /sec.
  - Extended network: scale distances by $\sqrt{n}$.
- Here we take $A = 1$ and $\lambda = n$: dense network
Scaling laws

Transport capacity

- Different schools of thought (both assume power law decay):
  - « Network theory »: interference = noise ($\gamma = 1$), node distribution is uniform on A (Gupta Kumar (IEEE Trans IT, 2000)).
  - Lower bounds.
  - Information theory: allow arbitrary operation and node placement (Xie Kumar IEEE Trans IT, 2004; Leveque Telatar IEEE Trans IT, 2005).
  - Upper bounds.

- Upper bound: $\Theta(n) = O\left(\frac{1}{\sqrt{n}}\right)$

- Lower bound: two approaches:
  - « Full connectivity first »: $\Theta(n) = \Omega\left(\frac{1}{\sqrt{n \log n}}\right)$
  - « Percolation »: $\Theta(n) = \Omega\left(\frac{1}{\sqrt{n}}\right)$

Upper bound (1)

- $n$ nodes distributed on a unit area ($\lambda = n$).
- Take $L(x) = x^{-\alpha}$
- For simplicity, take $P_i = P$ for all $i$ and $\beta > 1$.
- What is the maximal throughput $\Theta(n)$?
- We follow Gupta-Kumar 2000
- Suppose bit $b$ traverses $h(b)$ hops from source to destination, with $\delta_{h}^b$ denoting the distance traversed during the $h$th hop.
- Suppose that $n \Theta(n)$ bits are served all together during 1 sec.
- Let $L$ be the average distance of an OD pair: $L = O(1)$ for unit area.
- Total distance covered by all bits served in one second is

$$n \Theta(n) L \leq \sum_{h=1}^{n} \sum_{b=1}^{\Theta(n)} \delta_{h}^b$$
Upper bound (2)

- $n$ nodes distributed on a unit area ($\lambda = n$).
- Take $L(x) = x^{-\alpha}$.
- For simplicity, take $P_i = P$ for all $i$ and $\beta > 1$.
- Let $r(i)$ be the receiver from node $i$. Then

$$SNR_{i \rightarrow j} = \frac{P_i L(x_i)}{N_0 + \gamma \sum_{k \neq r(i)} P_k L(x_k)} \geq \beta$$

$$\Rightarrow P_i L(x_i) \geq \beta P_k L(x_k)$$

$$\Rightarrow \|x_k - x_{r(i)}\| \geq \beta^{1/\alpha} \|x_i - x_{r(i)}\|$$

Upper bound (3)

Using triangle inequality:

$$\|x_k - x_{r(i)}\| \geq \beta^{1/\alpha} \|x_i - x_{r(i)}\|$$

$$\|x_{r(i)} - x_{r(k)}\| \geq \|x_{r(k)} - x_k\| - \|x_{r(i)} - x_k\| \geq \beta^{1/\alpha} \|x_i - x_{r(i)}\| - \|x_{r(k)} - x_k\|$$

$$\|x_{r(i)} - x_{r(k)}\| \geq \|x_{r(k)} - x_i\| - \|x_{r(i)} - x_i\| \geq \beta^{1/\alpha} \|x_k - x_{r(k)}\| - \|x_{r(i)} - x_i\|$$

$$\|x_{r(i)} - x_{r(k)}\| \geq \frac{\beta^{1/\alpha} - 1}{2} (\|x_i - x_{r(i)}\| + \|x_k - x_{r(k)}\|)$$

- Each transmission consumes at least a circular footprint of radius $(\beta^{1/\alpha} - 1) *$ distance traveled in one hop.
- Packing constraint on unit area:

$$\sum_{b=1}^{n(n-1)/2} \pi \left( \frac{\beta^{1/\alpha} - 1}{2} \delta^\alpha_b \right)^2 \leq 1$$
Scaling laws

**Upper bound (4)**

- Let $H = \sum_b h(b)$ be the total number of hops traversed by all bits in 1 second ($H \leq n/2$)
- By convexity of quadratic function:
  \[
  \left( \sum_{b=1}^{n} \sum_{h=1}^{H} \frac{1}{H} \delta_{h}^{b} \right)^{2} \leq \sum_{b=1}^{n} \sum_{h=1}^{H} \frac{1}{H} \left( \delta_{h}^{b} \right)^{2} \leq \frac{1}{H \pi (\beta^{\gamma} - 1)^{2}}
  \]
- Combining all inequalities:
  \[
  n \theta(n)L \leq \frac{\sqrt{H}}{\sqrt{\pi (\beta^{\gamma} - 1)}} \leq \frac{\sqrt{n}}{\sqrt{2\pi (\beta^{\gamma} - 1)}}
  \]
  \[
  \theta(n) \leq \frac{1}{\sqrt{2\pi (\beta^{\gamma} - 1))L}} \frac{1}{\sqrt{n}}
  \]
  \[
  \rightarrow \Theta(n) = O \left( \frac{1}{\sqrt{\frac{1}{n}}} \right)
  \]

**Upper bound (5)**

- Shape of attenuation function $L(x)$ is important.
- Strict power law function $L(x) = x^{-\alpha}$:
  - Particular scaling properties! Near field effects are dominant for dense networks.
  - $\Theta(n) = O \left( \frac{1}{\sqrt{n}} \right)$ for extended and dense networks (Gupta Kumar (IEEE Trans IT, 2000)).
- Bounded attenuation function, ex: $L(x) = \min(1, x^{-\alpha})$:
  - $\Theta(n) = O \left( \frac{1}{\sqrt{n}} \right)$ for extended networks (Xie Kumar IEEE Trans IT, 2004)
  - $\Theta(n) = O \left( \frac{1}{n} \right)$ for dense networks (Dousse, Thiran, Infocom 2004)
Lower bound

- $n$ nodes on a $1 \times 1$ surface.
- Take $L(x) = x^{-\alpha}$ (more general attenuation functions can be considered).
- For simplicity, take $p_i = P$ for all $i$.
- What is the minimal throughput $\Theta(n)$?

Lower bound

- $n$ nodes on a $1 \times 1$ surface.
- What is the minimal throughput $\Theta(n)$?
- Two approaches:
  - « Full connectivity first »: first connect all nodes, then see how many transmissions can be scheduled. (Gupta Kumar (IEEE Trans IT, 2000); Kulkarni Viswanath (IEEE TIT 2002); El Gamal, Mammen, Prabhakar, Shah (Infocom 2004). We slightly adapt proof of Kulkarni Viswanath (IEEE TIT 2002).
  - « Percolation »: get the network supercritical, then connect isolated nodes (Franceschetti, Dousse, Tse, Thiran (ISIT 2004)).
- Other traffic matrices, hybrid networks:
  - Toumpis, Goldsmith (IEEE Trans. Wireless Communications, 2003), Liu, Liu, Towsley (Infocom 2003), ...
Full connectivity approach (1)

- Divide domain in squarelets of size $c_n = \sqrt{\log n / n}$.
- $P(\text{squarelet populated}) = 1 - \exp(-nc_n^2) = 1 - 1/n^2$
  -> All $(n/\log n)$ squarelets are populated w.h.p.

\[ c_n = \sqrt{\log n / n} \]

Full connectivity approach (2)

- Adjust power $P$ so that each node can transmit to any destination in one of the 4 neighboring squarelets.
- TDMA schedule (9 equivalence classes): 1 node per squarelet can transmit to any destination in one of the 4 neighboring nodes with a rate $R$ independent of $n$.

\[ c_n = \sqrt{\log n / n} \]
Full connectivity approach (2)

- TDMA schedule (9 equivalence classes): 1 node per squarelet can transmit to any destination in one of the 4 neighboring nodes with a rate $R$ independent of $n$.
- Interference from all nodes in 1 equivalence class, with $c_n = \sqrt{\frac{\log n}{n}}$.

$$I = \sum_{i} 8PL(3c_n(2i-1)) = 8P(3c_n)^{\alpha} \sum_{i} i(2i-1)^{\alpha} = KP(3c_n)^{\alpha}$$

$$SNR_{i\rightarrow j}(n) = \frac{PL\|x_i - x_j\|}{N_0 + \sum_{l=1, l \neq j} P_L\|x_l - x_j\|} \geq \frac{P(3c_n)^{\alpha}}{N_0 + I} \geq \frac{1}{N_0(3c_n)^{\alpha} / P + K}$$

$$R(n) = \frac{1}{2} \log(1 + SNR_{i\rightarrow j}(n)) \geq \frac{1}{2} \log\left(1 + \frac{1}{N_0(3c_n)^{\alpha} / P + K}\right) = R$$

Full connectivity approach (3)

- Slabs of size $1 \times c_n$ contains whp no more than $2 \sqrt{n \log n}$ nodes.

$c_n = \sqrt{\frac{\log n}{n}}$
Routing: use horizontal and then vertical paths.
Scheduling: \( 2 \times \text{Horiz/vertical} \times 9 \times \text{TDMA} \times \text{number of nodes in the slab whose traffic needs to be relayed} \leq 2 \sqrt{\frac{n \log n}{n}} \leq 36 \sqrt{\frac{n \log n}{n}} \)  
\( \Theta(n) = \Omega\left(\frac{1}{\sqrt{n \log n}}\right) \)

\[ c_n = \sqrt{\frac{\log n}{n}} \]

Percolation approach

\( n \text{ nodes on a 1 x 1 surface.} \)
High density = more disjoint paths

- subcritical
- supercritical
- fully connected

- Divide domain squarelets of size $c_n = c/\sqrt{n}$.
- $p = P(\text{squarelet populated}) = 1 - \exp(-nc_n^2) = 1 - \exp(-c^2)$.
- Pick $c > \sqrt{\log 2}$: many empty squarelets, but network supercritical.
Percolation approach (1)

- Note: in the following, should work with diagonal lattice to for bond percolation. Equivalent to working with squarelets, with $c$ rescaled by $\sqrt{2}$. For the sake of simplicity in exposure, we continue working here with the squarelets.

- Divide domain squarelets of size $c_n = c/\sqrt{n}$.
- Pick $c > \sqrt{(\log 2)}$: $p > \frac{1}{2}$
- There exist crossing paths of adjacent non empty squarelets.

Percolation approach (2)

- Divide domain squarelets of size $c_n = c/\sqrt{n}$.
- Pick $c > \sqrt{(\log 2)}$: $p > \frac{1}{2}$
- There exist crossing paths of adjacent non empty squarelets.
Scaling laws

Percolation approach (3)

- Divide domain squarelets of size $c_n = c/\sqrt{n}$, with $c > \sqrt{\log 2}$ ($p > \frac{1}{2}$).
- Take even $c$ a bit larger: there exist many crossing paths of adjacent full squarelets = highways.
- Use these paths as the "wireless backbone" to relay traffic.

Percolation approach (4)

- How many highways?
- Pick a rectangle $R_n$ of size $1 \times d_n = 1 \times c_n \log(1/c_n) = 1 \times (c/\sqrt{n}) \log(\sqrt{n}/c_n)$.
- Theorem (from Grimmett 1981): Let
  - $A_n$ be the event that there exist an open LR crossing of rectangle $R_n$.
  - $I_r(A_n)$ be the event that there are $r$ edge-disjoint open LR crossings of rectangle $R_n$.
  
  \[
  1 - P_p(I_r(A_n)) \leq \left(\frac{p}{p-r}ight)^r \left[1 - P_p'(A_n)\right]
  \]
**Percolation approach (5)**

- **Duality**
  - $A_n$ is the event that there exist an open LR crossing of rectangle $R_n$ of size $1 \times d_n$
  - $B_n$ is the event that there exist an closed TB crossing of dual of rectangle $R_n$
  - $P_p(A_n) + P_p(B_n) = 1$
  - $P_p(B_n) = P_{1-p}'(\text{open TB crossing of dual of rectangle } R_n) \\ \leq \frac{\sqrt{n/c}}{(4/3)(3(1-p'))^{dn}}$

- **Remember** $p = 1 - \exp(-c^2)$; pick $p' = 2p - 1$ and $c^2 > \log 6 + 2$.

- **Take** $r = b \log(n/c)$, with $b = 1 - (\log 6 + 2)/c^2$. Then

  $$1 - P_p(I_r(A_n)) \leq \left(\frac{p}{p-p'}\right)[1 - P_p'(A_n)]$$

  $$\leq e^{r^2}[P_p'(B_n)]$$

  as $n \to \infty$.

**Percolation approach (6)**

- **If** $c$ is large enough, there is a constant $b = b(c) > 0$ such that w.h.p. there are $r_n(c) = b \log(n/c)$ edge-disjoint open LR crossings in any rectangle $R_n$ of size $1 \times d_n = 1 \times (c/\sqrt{n}) \log(\sqrt{n/c})$

- **Highways are well scattered on the area.**
Scaling laws

Percolation approach (7)

- TDMA schedule (9 equivalence classes): 1 node per squarelet can transmit to any destination in one of the 4 neighboring nodes with a rate $R$ independent of $n$.
- Interference from all nodes in 1 equivalence class, with $c_n = c/\sqrt{n}$.

\[ I = \sum_i S_P L(3c_n(2i-1)) = 8P(3c_n)^{\alpha} \sum_i i(2i-1)^{-\alpha} = K P(3c_n)^{\alpha} \]

\[ \text{SNR}_{\text{int}}(n) = \frac{P L}{N_0 + \sum_{k \neq j} P L} \geq \frac{P(3c_n)^{\alpha}}{N_0 + I} = \frac{1}{N_0(3c_n)^{\alpha}/P + K} \]

\[ R(n) = \frac{1}{2} \log \left( 1 + \text{SNR}_{\text{int}}(n) \right) \geq \frac{1}{2} \log \left( 1 + \frac{1}{N_0(3c_n)^{\alpha}/P + K} \right) = R \]

Percolation approach (8)

- Each rectangle $R_n$ of size $1 \times d_n$ contains $r_n = b \log(\sqrt{n}/c)$ edge-disjoint open LR crossings w.h.p.
- Divide each rectangle $R_n$ in $r_n$ slabs $S_n$ of size $1 \times b_n$, with $b_n = d_n/r_n = (c/\sqrt{n}) \log(\sqrt{n}/c)$.
- Each slab $S_n$ contains w.h.p. no more than $2c/\sqrt{n}$ nodes.

\[ b_n = c_n/b \]
\[ d_n = c_n \log(1/c_n) \]
Scaling laws

**Percolation approach (9)**

- Routing on highways: use horizontal and then vertical paths.
- Scheduling: 2 (Horiz/vertical) x 9 (TDMA) x number of nodes in the slab whose traffic needs to be relayed ≤ 2c√n/b ≤ (36c/b)√n

→ Highway rate is Ω(1/√n)

\[ b_n = c_n/b \]

**Percolation approach (10)**

- Draining and delivery phases = 2 more slots for nodes not on highways
- Fresh traffic only, can make much longer hops

→ \( \Theta(n) = \Omega(1/\sqrt{n}) \)
Percolation approach

- Use the well scattered highways as "wireless backbone" to relay traffic
- TDMA for transmissions along the highways.
- Rate per node on the highway of $\Omega(1/n)$ is achievable.

Throughput per node vs Range

- No interference ideal network
- Interference limited network

$\lambda = \lambda_c \quad \lambda = \lambda^* \quad \lambda = \log n \quad \lambda = n$
Content

1. From Boolean model to STIRG (Signal To Interference Ratio Graph)
2. Transport capacity a la Gupta-Kumar (interference = foe)
   - Upper bound
   - Lower bound: full connectivity vs percolation
3. Relay channel (interference = friend)
   - Lower bound: full connectivity vs percolation
Information theoretic relay model (interference = friend)

- Nodes \(i\) and \(j\) are connected if it is possible to exchange data between them at rate at least \(R\), possibly using all other nodes as relays (relay channel).
- This definition assumes the knowledge of many parameters (maximal power \(P\), ambient noise \(N_0\)). Here we consider them all fixed, except the rate \(R\).

Full connectivity

- Nodes distributed as a Poisson process of unit intensity \(\lambda\) on a square of surface \(A\).
- Attenuation function \(L(d)\) of the form \(d^{-\alpha}\), with \(\alpha > 1\).
- Every node must be connected at rate at least \(R\) under the relay model to an arbitrarily chosen node (say 0).
- Theorem 5 (Liu & Srikant)
  - if \(R(A) = \Omega((\log A)^{-\alpha})\), the network is disconnected w.h.p when \(A \to \infty\)
  - if \(R(A) = O((\log A)^{-\alpha})\), then rate \(R(A)\) is achievable between any two nodes w.h.p when \(A \to \infty\)
- A rather pessimistic result since \(R \to 0\) as \(A \to \infty\)
- Gastpar, Vetterli (Infocom 2002): dense network case (\(\lambda \to \infty\)), \(R > 0\) if dead zone without any node around the source \(i\) and destination \(j\).
Partial connectivity

- Nodes distributed as a Poisson process of unit intensity $\lambda$ on a square of surface $A$.
- Attenuation function $L(d)$ of the form $d^{-\alpha}$, with $\alpha > 2$.
- A fraction $\Theta$ of nodes must be connected at rate at least $R$ under the relay model to an arbitrarily chosen node $(0)$.
- Theorem 6 (Dousse, Franceschetti & Thiran (Infocom'05))
  - For any $0 < \Theta < 1$, there exists a rate $R > 0$ independent of the network size, such that a fraction at least $\Theta$ of the nodes can exchange data at rate at least $R$.
  - For any $R > 0$, the fraction of nodes that can exchange data at rate $R$ is at most $\Theta_u \text{w.h.p.}$, where
    \[ \Theta_u = \Pr(S(0) \geq N(\exp(2R)-1)/P) < 1 \]
    \[ S(0) = \sum_{\text{nodes at location } x} L^2(||x||) \]
- A rather optimistic result since $R > 0$ as $A \to \infty$ when we drop a tiny fraction of very isolated nodes.

Partial connectivity

- Theorem 6 (Lower bound for the relay channel)
  - For any $0 < \Theta < 1$, there exists a rate $R > 0$ independent of the network size, such that a fraction at least $\Theta$ of the nodes can exchange data at rate at least $R$.
  - Proof: Let $r$ be the radius such that in the Boolean model, the fraction of nodes connected to the destination at the origin is at least $\Theta$.
    Construct a TDMA scheme with 3 time-slots such that any node in the cluster containing the origin can exchange data with a rate at least
    \[ R = \log(1 + PL(r)/(N_0 + 6P\sum_k k L(kr))) / 8 \]
Partial connectivity

Theorem 6 (Upper bound for the relay channel)
For any $R > 0$, the fraction of nodes that can exchange data at rate $R$ is at most $\Theta_u$ w.h.p., where
\[
\Theta_u = P \left( S(0) \geq N_0 (\exp(2R) - 1) / P \right) < 1
\]
and $S(0) = \sum_{x_i \neq 0} L^2(||x_i||)$.

Proof:
(1) We use 2 results of information theory:
- Max-flow min-cut theorem, $R \leq$ rate of multiple receiver Gaussian channel, with 1 source at $x_k$ and destinations at all other nodes at locations $x_i, i \neq k$)
- Rate of multiple receiver Gaussian channel is known (Telatar 1999):
  \[
  R \leq \log(1 + P \sum_{i \neq k} L^2(||x_i - x_k||) / N_0) / 2
  \]
so one must have $S(x_k) = \sum_{i \neq k} L^2(||x_i - x_k||) \geq N_0 (\exp(2R) - 1) / P := M$.

(2) Compute the fraction of points $k$ such that $S(x_k) \geq M$. Using stationarity and mixing arguments, find that this fraction is
\[
P(S(0) \geq M) = \Theta_u.
\]
Partial connectivity

- Fraction nodes connected under the relay channel model with rate R.

![Graph showing partial connectivity](image)

Conclusion

- Percolation theory is very useful for sensor and ad hoc networks.
- Physical model (STIRG): Percolation occurs despite interferences.
- Proof for $L(x)$ with a compact support. Not a necessary condition (current work with N. Macris).
- Critical curve $\gamma(\lambda)$ is non-monotone.
  - Proved that $\gamma(\lambda)$ decreases as $O(1/\lambda)$ for $\lambda \to \infty$.
  - Proved that $\gamma(\lambda) > 0$ if $\lambda > \lambda^*$. (current work with M. Franceschetti and R. Meester)
- Simple TDMA performs as well as CDMA.
- Transport capacity: Percolation allows to bridge the gap in the Gupta-Kumar model (Throughput $\sim 1/\sqrt{n}$ per node for uniform traffic matrix).
- Information theoretic relay model: Requiring partial connectivity instead of full connectivity is necessary and sufficient to have non-zero communication rate $R$ for asymptotically large networks.
- References on line at [www.mics.org](http://www.mics.org)
Thanks

- Olivier Dousse (EPFL) - PhD thesis available on line
- Connectivity and STIRG: François Baccelli (ENS-INRIA). More recent work with Massimo Franceschetti, Nicolas Macris and Ronald Meester.
- Transport capacity: Massimo Franceschetti (UCSD) and David Tse (UC Berkeley).
- Not in this tutorial:
  - Latency and first passage percolation (with Olivier Dousse and Petteri Mannersalo (VTT Finland))
  - Detection of intruder (with Olivier Dousse and Christina Tavoularis (Cornelle)).