Rules for the exam. It is forbidden to use notes, books or calculators. When needed, use approximations. Report the solution of each problem on a different sheet of paper.

Time available: 70 minutes.

PROBLEM A
Design a rearrangeable switch of size $900 \times 450$ using only modules of size $10 \times 10$, with the aim of minimizing the number of modules.

1. Describe the architecture
2. Compute the total number of modules required
3. Describe the configuration algorithm
4. Write the formula to compute the minimum theoretical number of modules to build the switch and to compare the actual complexity to the optimal one

PROBLEM B
Consider a $4 \times 4$ input queued switch. Report four examples, one for each below listed case, of the queue occupancies such that:

1. the maximum size matching is different from the maximal size matching
2. the maximum weight matching is different from the maximal weight matching
3. the maximum size matching is different from the maximum weight matching
4. the maximal size matching is different from the maximal weight matching

PROBLEM C
Consider a generic bufferless $N \times M$ switch with synchronous operation: at most one cell is transferred for each timeslot from each input and to each output. The traffic is multicast and uniformly distributed among all inputs and the fanout set is uniformly distributed among all the outputs. Fanout splitting is allowed, i.e. a cell can be transferred in one timeslot to a subset of its destinations.

Let $\rho$ be the arrival probability of a cell at each input during one timeslot. Given a multicast cell, let $q$ be the probability that the cell is directed to a specific output.

1. What is the distribution of the fanout (i.e. the number of destinations) of a generic cell?
2. What is the average fanout $f$ of a cell?
3. What is the average offered load at each output?
4. Under which conditions the traffic is admissible?

Now fix the attention to a specific output.

1. What is the distribution of the number of cells directed to that specific output?
2. What is the average throughput as a function of $\rho$ and $f$?
3. What is the maximum throughput under admissible traffic for finite $N$ and $M$? What about taking the limits for $N$ and $M$ going to infinity?
SOLUTIONS

Problem A
The $900 \times 450$ switch can be built using a Clos network in the following way:

$$C_{900 \times 450} = 90C_{10} + 10C_{90 \times 45} + 45C_{10}$$

where the $90 \times 45$ switch can be also built using a Clos network:

$$C_{90 \times 45} = 9C_{10} + 10C_{9 \times 5} + 45C_{10}$$

in which the last module of the last stage has 5 unconnected outputs. Now observe that a $9 \times 5$ switch can be built with a $10 \times 10$ module; hence

$$C_{90 \times 45} = 24C_{10}$$

and finally

$$C_{900 \times 450} = 375C_{10}$$

Paul’s algorithm is used to configure the network, and should be applied recursively twice.

Now the total number $S$ of possible states of the whole switching network is:

$$S = (10!)^{375}$$

whereas the total number of configurations $X$ is

$$X = \frac{900!}{450!}$$

Hence, the average number of states for each configuration is

$$\frac{S}{X} = e^{\log S - \log X} = e^{375 \log(10!) - \log(900!) + \log(450!)}$$

Note that, with a calculator, this formula can be computed using the log-gamma function available in many numerical solvers: $\log(x + 1) = \log(x!)$. 

$$\frac{S}{X} = e^{375 \log(11) - \log(901) + \log(451)} \approx e^{2741}$$

Problem B
Let $R$ be the request matrix, corresponding to the VOQ occupancy. Let $MWM/mWM$ be the maximum/maximal weight matching, and $MSM/mSM$ be the maximum/maximal size matching. Given a matching $\pi$, let $\pi(i)$ be the output connected to input $i$; if $i$ is not connected, $\pi(i) = -$ .

1. $R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$: $\pi_{MSM} = (1, 2, 3, 4), \pi_{mSM} = (2, -3, 4)$.

2. $R = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$: $\pi_{MWM} = (1, 2, 3, 4), \pi_{mWM} = (2, -3, 4)$.

3. $R = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$: $\pi_{MSM} = (1, 2, 3, 4), \pi_{MWM} = (2, -3, 4)$.

4. $R = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$: $\pi_{mSM} = (1, 2, 3, 4), \pi_{MWM} = (2, -3, 4)$.

Problem C
Let $X$ be the fanout of a generic cell

$$P(X = x) = \binom{M}{x} q^x (1 - q)^{M-x} \quad 0 \leq x \leq M$$

Note that the model implies that, with probability $(1 - q)^M$, the packet fanout is null. In theory, $P(X = x)$ could be modified to avoid this case, but we have preferred to keep this case for the sake of simpler theory. Now, the average fanout is $f = E[X] = qM$ and the traffic is admissible if

$$N\rho f/M < 1 \quad \Rightarrow \quad N\rho q < 1$$
Let $Y$ be the number of cells directed for a specific output. Since $\rho q$ is the probability that an input has a cell destined to a specific output:

$$P(Y = y) = \binom{N}{y}(\rho q)^y(1 - \rho q)^{N-y} \quad 0 \leq y \leq N$$

The probability that no cell is received for an output is:

$$P(Y = 0) = (1 - \rho q)^N$$

The average throughput, seen at any output, is equal to the probability that an output is busy

$$T = P(Y > 0) = 1 - P(Y = 0) = 1 - (1 - \rho q)^N = 1 - \left(1 - \rho \frac{f}{M}\right)^N$$

Note that the maximum throughput is achieved for $\rho = 1/(qN)$:

$$T_{max} = 1 - \left(1 - \frac{1}{N}\right)^N$$

which goes to $1 - e^{-1} \approx 63\%$, independently of $M$, for $N \to \infty$. 