Capacity Scaling of Wireless Networks with Inhomogeneous Node Density: Upper Bounds

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Abstract

We analyze the capacity scaling laws of wireless ad hoc networks comprising significant inhomogeneities in the node spatial distribution over the network area. In particular, we consider nodes placed according to a shot-noise Cox process, which allows to model the clustering behavior usually recognized in large-scale systems. For this class of networks, we introduce novel techniques to compute upper bounds to the available per-flow throughput as the number of nodes tends to infinity, which are tight in the case of interference limited systems.

I. Introduction and related work

Wireless ad-hoc networks have been traditionally modelled as a set of nodes placed over a finite bi-dimensional domain and communicating among them (possibly in a multi-hop fashion) over point-to-point wireless links subject to mutual interference. In their seminal work, Gupta and Kumar [1] have considered a model in which n static nodes are placed in a disk of unit area, and establish n source-destination flows. For arbitrary network topologies, they obtain that the per-flow throughput is $O(1/\sqrt{n})$, thus providing an upper bound to the performance achievable under any node placement. In the case of nodes uniformly distributed over the area, they propose a scheme achieving $\Theta(1/\sqrt{n \log n})$ per-node throughput. Later on, Franceschetti at al. [2] have applied percolation theory results to show that $\Theta(1/\sqrt{n})$ transmission rate is achievable by the flows also under the uniform node distribution.

The goal of our work is to extend the capacity scaling analysis to networks exhibiting a much higher variability in the node spatial distribution than the one resulting from a homogeneous

Poisson point process. Indeed, almost all large-scale structures created by human or natural processes over geographical distances (such as urban or sub-urban settlements) are characterized by significant degrees of clustering, due to spontaneous grouping of the nodes around a few attraction points. This motivated us to considering a general class of clustered point processes referred to as shot-noise Cox processes [3], which includes several special cases widely used in many different fields, such as Neyman-Scott process [4], Matérn cluster process [5], Thomas process [6].

While analyzing the impact on the network capacity of such node placement processes, we maintain the basic assumptions originally introduced by Gupta and Kumar, and derive upper bounds to the per-flow throughput as the number of nodes (and the number of clusters) tends to infinity. Our main finding is that the network capacity is intrinsically related to the minimum intensity of the overall point process over the area. In a separate paper [7], we introduce a class of scheduling and routing schemes that approach the limits presented here (up to a poly-log factor), thus showing that our upper bounds are tight for the case in which the system performance is limited by interference among concurrent transmissions.

To the best of our knowledge, only a few works have analyzed the capacity of clustered networks. In [8], Toumpis considers a set of n nodes wishing to communicate to $m = \Theta(n^d)$ cluster heads $(0 < d < 1)$, and discovers that the network throughput can be limited by the formation of bottlenecks at the clusters heads. Both sources and cluster heads are uniformly distributed, so the overall node density does not exhibit inhomogeneities.

The deterministic approach proposed in [9] allows to derive capacity results also for some non-i.i.d. node distributions. In particular, the authors consider nodes distributed over $\sqrt{n}$ lines, or clustered around $\sqrt{n}$ neighborhoods. In both cases, a regular square tessellation of the network area can be built in such a way that no squarelet is empty w.h.p., while the maximum number of nodes in each squarelet increases at most as a poly-log function of n. Therefore, the network does not contain significant inhomogeneities, and the resulting capacity is similar to that derived by Gupta and Kumar.

In [10] the authors consider a system which contains many circular clusters with uniform node density within them, whose centres are distributed according to a Poisson process over the network area (a Matérn cluster process). Moreover, clusters are surrounded by a sea of nodes with much lower node density. The only quantity that scales with n is the network size. Below

a critical network size, the per-node throughput is limited by the amount of data that a cluster can exchange with the sea of nodes, whereas above the critical size the per-node throughput is limited by the capacity of the sea of nodes. In contrast to [10], we consider a more general shot-noise Cox process, and we let the density of clusters (and the number of nodes per cluster) to scale with n as well. Moreover our techniques to compute upper bounds to the capacity are totally different.

The authors of [11] consider nodes placed according to Poisson cluster processes similar to ours, and focus their attention on a specific transmitter-receiver pair at a distance R apart. They characterize the distributional properties of the interference at the receiver, and the outage probability under Rayleigh fading.

In [12] the authors present a spatial framework to upper bound the number of simultaneous transmissions in a network with general topology. However, it is unclear how their approach can be used to upper bound the capacity of the class of networks considered here, and the corresponding per-flow throughput.

II. System Assumptions and Notation

A. Network Topology

We consider networks composed of a random number N of nodes (being $E[N] = n$) distributed over a square region $\mathcal{O}$ of edge L. To avoid border effects, we consider wrap-around conditions at the network edges (i.e., the network area is assumed to be the surface of a two-dimensional Torus). The network physical extension L is allowed to scale with the average number of nodes, since this is expected to occur in many growing systems. Throughout this work we will always assume that $L = \Theta(n^{\alpha})$, with $\alpha \geq 0$.

The clustering behavior of large scale systems is taken into account assuming that nodes are placed according to a shot-noise Cox process (SNCP). An SNCP can be conveniently described by the following construction. We first specify a point process $\mathcal{C}$ of cluster centres, whose positions are denoted by $\mathbf{C} = \{c_j\}_{j=1}^{M}$, where M is a random number with average $E[M] = m$. In the literature the centre points c_j are also called parent or mother points. Each centre point c_j in turn generates a point process of nodes whose intensity at ξ is given by $q_j k(c_j, \xi)$, where $q_j \in (0, \infty)$ and $k(c_j, \cdot)$ is a dispersion density function, also called kernel, or shot. In the literature the nodes generated by each centre are referred to as offspring or daughter points. The

overall node process $\mathcal{N}$ is then given by the superposition of the individual processes generated by the cluster centres. The conditional local intensity at ξ of the resulting SNCP is

$$\Phi(\xi) = \sum_j q_j k(c_j, \xi)$$

Notice that $\Phi(\xi)$ is a random field, in the sense that, conditionally over all (q_j, c_j), the node process $\mathcal{N}$ is an (inhomogeneous) Poisson point process with intensity function Φ. We denote by $\mathbf{X} = \{X_i\}_{i=1}^N$ the collection of nodes positions in a given realization of the SNCP.

In this work we restrict ourselves to kernels $k(c_j, \cdot)$ which are invariant under both translation and rotation, i.e., $k(c_j, \xi) = k(\|\xi - c_j\|)$ depends only on the euclidean distance $\|\xi - c_j\|$ of point ξ from the cluster centre c_j. Moreover we assume that $k(c_j, \cdot)$ is a summable, non-increasing, bounded and continuous function whose integral $\int_{\mathcal{O}} k(c_j, \xi)\, \mathrm{d}\xi$ over the entire network area is finite and equal to 1. In practice, the kernels considered in our work can be specified by first defining a non-increasing, bounded and continuous function $s(\rho)$ such that $\int_0^\infty \rho\ s(\rho)\, \mathrm{d}\rho < \infty$ [1] and then normalizing it over the network area $\mathcal{O}$:

$$k(c_j, \xi) = \frac{s(\|\xi - c_j\|)}{\int_{\mathcal{O}} s(\|\zeta - c_j\|)\, \mathrm{d}\zeta}$$

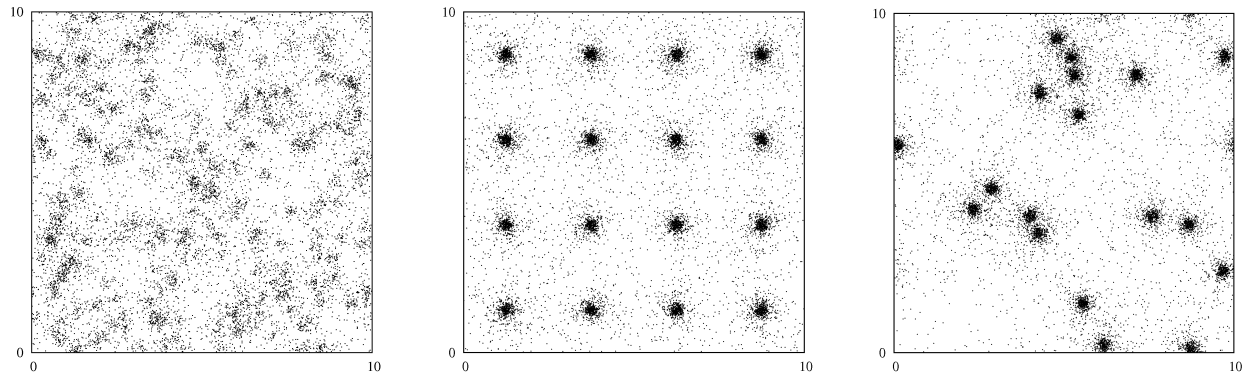
Notice that in our asymptotic analysis[2] we can neglect the normalizing factor $\int_{\mathcal{O}} s(\|\zeta - c_j\|)\, \mathrm{d}\zeta = \Theta(1)$. Moreover $k(c_j, \xi) = \Theta(s(\|\xi - c_j\|))$. At last, we observe that, in order to have finite integral over increasing networks areas, functions $s(\rho)$ must be $o(\rho^{-2})$, i.e., they must have a tail that decays with the distance faster than quadratically.

Under the above assumptions on the kernel shape, the quantity q_j equals the average number of nodes generated by cluster centre c_j. We assume that all cluster centres generate on average the same number of nodes, hence $q_j = q = n/m$. In our work, we let q scale with n as well (clusters are expected to grow in size as the number of nodes increases). This is achieved assuming that the average number of cluster centres scales as $m = \Theta(n^\nu)$, with $\nu \in (0, 1]$. Consequently, the average number of nodes per cluster scales as $q = \Theta(n^{1-\nu})$.

At last we need to specify the point process $\mathcal{C}$ of cluster centres. We consider two different models:

[1] In the case $\alpha = 0$ the condition $\int_0^\infty \rho\ s(\rho)\, \mathrm{d}\rho < \infty$ can be relaxed to $\int_0^L \rho\ s(\rho)\, \mathrm{d}\rho < C$ for some $C > 0$.

[2] Given two functions $f(n) \geq 0$ and $g(n) \geq 0$: $f(n) = o(g(n))$ means $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = O(g(n))$ means $\limsup_{n\to\infty} f(n)/g(n) = c < \infty$; $f(n) = \omega(g(n))$ is equivalent to $g(n) = o(f(n))$; $f(n) = \Omega(g(n))$ is equivalent to $g(n) = O(f(n))$; $f(n) = \Theta(g(n))$ means $f(n) = O(g(n))$ and $g(n) = O(f(n))$; at last $f(n) \sim g(n)$ means $\lim_{n\to\infty} f(n)/g(n) = 1$.



(a) Cluster Random model with $\nu = 0.6$.

(b) Cluster Grid model with $\nu = 0.3$.

(c) Cluster Random model with $\nu = 0.3$.

Fig. 1. Examples of topologies comprising $n = 10,000$ nodes distributed over the square 10×10 ($\alpha = 0.25$). In all three cases $s(\rho) \sim \rho^{-2.5}$. Case 1(a) belongs to the *cluster-dense* regime ($\alpha < \nu/2$). Cases 1(b) and 1(c) belong to the *cluster-sparse* regime ($\alpha > \nu/2$).

Cluster Grid Model. Clusters centres are placed in a deterministic fashion over the vertices of a regular square grid covering the network area $\mathcal{O}$.

Cluster Random Model. Cluster centres are randomly placed according to a Homogeneous Poisson Process (HPP) of intensity $\phi_c = m/L^2$ over $\mathcal{O}$.

The Cluster Grid Model is simpler to analyze, because the overall node process turns out to be a standard Inhomogeneous Poisson Process (IPP) whose intensity over the area can easily be computed, since the clusters centres $\mathbf{C}$ are assigned. This model serves as an intermediate step towards the analysis of the more complex Cluster Random Model.

For both models, we define $d_c = L/\sqrt{m} = \Theta(n^{\alpha-\nu/2})$. This quantity represents, in the case of the Cluster Grid Model, the distance between two neighboring cluster centres on the grid; in the case of the Cluster Random Model, d_c is the edge of the square where the expected number of cluster centres falling in it is equals to 1. We call *cluster-dense* regime the case $\alpha < \nu/2$, in which d_c tends to zero an n increases. We call *cluster-sparse* regime the case $\alpha > \nu/2$, in which d_c tends to infinity an n increases.

Figure 1 shows three examples of the kind of topologies considered in this paper, in the case of $n = 10,000$ and $\alpha = 0.25$. In all three cases we have assumed $s(\rho) \sim \rho^{-2.5}$.

B. Communication Model

We assume that time is divided into slots of equal duration, and that in each slot an optimal scheduling policy enables a set of transmitter-receiver pairs to communicate over point-to-point wireless links which are modelled as Gaussian channels of unit bandwidth. We consider point-to-point coding and decoding, hence signals received from nodes other than the (unique) transmitter are regarded as noise. We remark that this is not the most general setting from an information theory point of view, as nodes could potentially employ cooperative coding-decoding schemes.

We assume that interference among simultaneous transmissions is described by the so called *generalized physical model*, according to which the rate achievable by node i transmitting to node j in a given time slot is limited to

$$R_{ij} = \log_2(1 + \text{SINR}_j)$$

where SINR_j is the signal to interference and noise ratio at receiver j:

$$\text{SINR}_j = \frac{P_i \ell_{ij}}{N_0 + \sum_{k \in \Delta, k \neq i} P_k \ell_{kj}}$$

Here, Δ is the set of nodes which are enabled to transmit in the given slot, P_i is the power emitted by node i, ℓ_{ij} is the power attenuation between i and j, and N_0 is the ambient noise power. The power attenuation is assumed to be a deterministic function of the distance d_{ij} between i and j, according to $\ell_{ij} = d_{ij}^{-\gamma}$, with $\gamma > 2$. One drawback of this model is that the received power (and the corresponding rate) is amplified to unrealistic levels when d_{ij} tends to zero. Some authors have suggested to account for near-field propagation effect by bounding the attenuation function to 1: $\ell_{ij} = \min\{1, d_{ij}^{-\gamma}\}$. However, any fixed bound leads to pathological throughput degradation in network regions where the node density tends to infinity, as pointed out in [13]. To avoid such problems, we simply assume that the achievable rate on any link cannot grow arbitrarily large, but is bounded by a constant R_0 due to physical limitations of transmitters/receivers (maximum data speed of I/O devices, finite set of possible modulation schemes, etc). Therefore we consider the following variant of the *generalized physical model*:

$$R_{ij} = \min\{R_0, \log_2(1 + \text{SINR}_j)\}$$

while keeping $\ell_{ij} = d_{ij}^{-\gamma}$, for any d_{ij}.

We assume that nodes can employ different transmitting powers, according to an optimal strategy of power assignment to simultaneous transmissions.

C. Traffic Model

Similarly to previous work we focus on *permutation traffic patterns*, i.e., traffic patterns according to which every node is source and destination of a single data flow at rate λ. Sources and destinations of data flows are randomly matched, establishing N end-to-end flows in the network. Note that a permutation traffic pattern is represented by a traffic matrix of the form $\mathbf{\Lambda} = \lambda\hat{\mathbf{\Lambda}}$ being $\hat{\mathbf{\Lambda}}$ a permutation matrix (i.e., a binary valued doubly stochastic matrix).

Let $B(t)$ be the network backlog, that is, the number of data units already generated by sources which have not yet been delivered to destinations at time t. We say that traffic $\lambda\hat{\mathbf{\Lambda}}$ is *sustainable* if there exists a scheduling-routing policy such that $\limsup_{t\to\infty} B(t)/t = 0$ w.p.1.

D. Asymptotic Analysis of the Capacity

As the average number of nodes increases, we generate a sequence of systems indexed by n. To summarize, the quantities that depend on n are: i) the network physical extension $L = n^{\alpha}$; ii) the number of cluster centres $m = n^{\nu}$, and consequently the average number of nodes belonging to the same cluster $q = n^{1-\nu}$. We are essentially interested in establishing how the network capacity scales with n under the assumptions we have introduced above on network topology, communication model and traffic pattern. The per-node capacity is $\Theta(h(n))$ if, given a sequence of random permutation traffic patterns with rate $\lambda = h(n)$, there exist two constants c, c' such that $c < c'$ and both the following properties hold:

$$\begin{cases} \lim_{n\to\infty} \Pr\{c\lambda^{(n)} \text{ is sustainable}\} = 1 \\ \lim_{n\to\infty} \Pr\{c'\lambda^{(n)} \text{ is sustainable}\} < 1 \end{cases}$$

Equivalently, we say in this case that the *network capacity* (or maximum network throughput) is $\Theta(n\,h(n))$.

To facilitate the reader, we have reported in Table I a collection of system parameters frequently used in the rest of the paper.

III. Asymptotic analysis of the SNCP

In this section we characterize the asymptotic behavior of the local intensity $\Phi(\xi)$ resulting from the considered shot-noise Cox process of node placement over the area. Recall that the conditional local intensity of nodes at point ξ can be written as $\Phi(\xi) = \sum_j q\,k(c_j, \xi)$. For both

Symbol	Definition
n	Average number of nodes
L	Edge length of the network area
α	Growth exponent of L: $L = \Theta(n^{\alpha})$, $\alpha \geq 0$
m	Average number of cluster centres
ν	Growth exponent of m : $m = \Theta(n^{\nu})$, $0 < \nu \leq 1$
q	Average number of nodes per cluster, $q = \Theta(n^{1-\nu})$
ϕ_c	Density of clusters centres over the area, $\phi_c = m/L^2$
d_c	Typical distance between cluster centres, $d_c = \Theta(n^{\alpha-\nu/2})$
$s(\rho)$	Radial shape of rotationally invariant kernel
λ	Per-flow throughput

TABLE I

SYSTEM PARAMETERS

Cluster Grid and Cluster Random models, we define the following two quantities: $\overline{\Phi} = \sup_{\mathcal{O}} \Phi(\xi)$ and $\underline{\Phi} = \inf_{\mathcal{O}} \Phi(\xi)$, denoting, respectively, the maximum and the minimum of $\Phi(\xi)$ over $\mathcal{O}$.

A. Cluster Grid Model

In the Cluster Grid model, $\overline{\Phi}$ and $\underline{\Phi}$ are two deterministic values depending only on the system parameters α, ν, $s(\cdot)$, since cluster centres in this case are regularly placed over the area, so that nodes are positioned according to an IPP with spatial (deterministic) intensity $\Phi(\xi)$. Recall that $d_c = \Theta(n^{\alpha-\nu/2})$ is the separation of cluster centres over the grid. It is of rather immediate verification, that, whenever $d_c = O(1)$, we have $\underline{\Phi} = \Theta(\overline{\Phi}) = \Theta(\frac{n}{L^2})$. When $d_c = \omega(1)$, $\underline{\Phi} = o(\overline{\Phi})$, being $\underline{\Phi} = \Theta(q\, s(d_c))$ and $\overline{\Phi} = \Theta(q)$.

B. Cluster Random Model

Things are slightly more involved in the Cluster Random model, in this case h both $\overline{\Phi}$ and $\underline{\Phi}$ are random variables which depend also on the positions $\mathbf{C}$ of the cluster centres, which are distributed according to an HPP of rate $\phi_c = m/L^2$. We will need the following lemma, which has been widely used in previous work:

Lemma 1: Consider a set of points $\mathbf{C}$ distributed over $\mathcal{O}$ according to an HPP at rate $\phi_c = m/L^2$. Let $\mathcal{A}$ be a regular tessellation of $\mathcal{O}$ (or any sub-region of $\mathcal{O}$), whose tiles A_k

have a surface $|A_k|$ non smaller than $16\frac{\log m}{\phi_c}$, $\forall k$. Let $U(A_k)$ be the number of points of $\mathbf{C}$ falling in A_k. Then, uniformly over the tessellation, $U(A_k)$ is comprised w.h.p. between $\frac{\phi_c|A_k|}{2}$ and $2\phi_c|A_k|$, i.e., $\frac{\phi_c|A_k|}{2} < \inf_k U(A_k) \leq \sup_k U(A_k) < 2\phi_c|A_k|$.

We do not repeat the proof of this lemma, which is based on a standard application of the Chernoff bound (see for example [14]).

Corollary 1: As immediate consequence of lemma 1, if we consider a regular tessellation in which $|A_k| = O(\log m/\phi_c)$, $\forall k$, then uniformly over the tessellation $U(A_k) = O(\log m)$.

The following theorem characterizes the extreme values of the local intensity as function of $d_c = L/\sqrt{m}$.

Theorem 1: Consider nodes distributed according to a Cluster Random model. Let $\eta(m) = d_c\sqrt{\log m}$. If $\eta(m) = o(1)$, then it is possible to find two positive constants g, G, with $g < G$, such that $\forall \xi_0 \in \mathcal{O}$,

$$g\frac{n}{L^2} < \Phi(\xi_0) < G\frac{n}{L^2} \qquad \text{w.h.p.} \tag{1}$$

which means that $\underline{\Phi} = \Theta(\overline{\Phi})$. More in general, it results $\overline{\Phi} = O(q\,\log m)$ and $\underline{\Phi} = \Omega(q \log m\, s(d_c\sqrt{\log m}))$.

Proof: The main steps of the proof are: i) the domain $\mathcal{O}$ is divided into squarelets; i) the local intensity at ξ_0 is expressed as sum of contributions, each due to cluster centres located in the same squarelet; ii) applying Lemma 1 every contribution is bounded w.h.p. (both from below and from above); iii) the upper (lower) bound is shown to converge w.h.p. to some value for $n \to \infty$.

More in details, consider a generic point $\xi_0 \in \mathcal{O}$. By definition:

$$\Phi(\xi_0) = \sum_{j=1}^{M} q\, k(\xi_0, c_j) = \sum_{j=1}^{M} q \frac{s(||\xi_0 - c_j||)}{\int_{\mathcal{O}} s(||\zeta - c_j||)\,\mathrm{d}\zeta}$$

Now, let $\mathcal{A}$ denote a regular square tessellation of $\mathcal{O}$, such that each squarelet A_k has area $|A_k| = 16\,\eta^2(m)$. Let $\underline{d}_{0k}$ and $\overline{d}_{0k}$ be, respectively, the inferior and the superior of the distances between points $\xi \in A_k$ and ξ_0, i.e., $\underline{d}_{0k} = \inf_{\xi \in A_k} \|\xi - \xi_0\|$ and $\overline{d}_{0k} = \sup_{\xi \in A_k} \|\xi - \xi_0\|$; at last, let $\underline{U}(A_k)$ and $\overline{U}(A_k)$ be, respectively, a lower bound and an upper bound to the number of cluster centres falling in A_k. It results:

$$\sum_k \frac{q}{H} s(\overline{d}_{0k})\underline{U}(A_k) < \Phi(\xi_0) < \sum_k \frac{q}{H} s(\underline{d}_{0k})\overline{U}(A_k)$$

being $H = \int_{\mathcal{O}} s(||\zeta - c_j||)\,\mathrm{d}\zeta$.

Applying Lemma 1 we have that, w.h.p., uniformly over k, $\underline{U}(A_k) \geq m/(2L^2)|A_k|$ and $\overline{U}(A_k) \leq 2m/L^2|A_k|$. Moreover, we observe that i) $\sum_k \frac{q}{H} s(\overline{d}_{0k})|A_k|$ and $\sum_k \frac{q}{H} s(\underline{d}_{0k})|A_k|$ can be interpreted, respectively, as lower Riemann sum and upper Riemann sum of $\int_{\mathcal{O}} \frac{q}{H} s(\|\xi - \xi_0\|)\, \mathrm{d}\xi$; ii) since $\eta(m) = o(1)$, the mesh size of the partitions associated to Riemann sums vanishes to 0 as $n \to \infty$. As a consequence:

$$\sum_k \frac{q}{H} s(\overline{d}_{0k})|A_k| \sim \sum_k \frac{q}{H} s(\underline{d}_{0k})|A_k| \sim \frac{q}{H} \int_{\mathcal{O}} s(\|\xi - \xi_0\|)\, \mathrm{d}\xi = q = \frac{n}{m}$$

and we conclude that:

$$\frac{n}{2L^2} = q\frac{m}{2L^2} < \Phi(\xi_0) < q\frac{2m}{L^2} = \frac{2n}{L^2}$$

Thus (1) is verified for any $0 < g \leq 1/2$ and $G \geq 2$.

When $\eta(m) = \Omega(1)$, $\sum_k \frac{q}{H} s(\underline{d}_{0k})\overline{U}(A_k)$ and $\sum_k \frac{q}{H} s(\overline{d}_{0k})\underline{U}(A_k)$ provide, respectively, an upper bound and a lower bound to the local intensity. It turns out: $\sum_k \frac{q}{H} s(\underline{d}_{0k})\overline{U}(A_k) = O(q \log m)$ and $\sum_k \frac{q}{H} s(\overline{d}_{0k})\underline{U}(A_k) = \Omega(q \log m\, s(d_c\sqrt{\log m}))$ ■

The above results show that, for both Cluster Grid and Cluster Random models, $\underline{\Phi} = \Theta(\overline{\Phi})$ in the *cluster-dense* regime ($\alpha < \nu/2$), whereas $\underline{\Phi} = o(\overline{\Phi})$ in the *cluster-sparse* regime ($\alpha > \nu/2$). In the following we will focus on the most interesting case in which $\underline{\Phi} = o(\overline{\Phi})$, considering separately the Cluster Grid model in Section IV and the Cluster Random model in Section V. In Section VI we will briefly discuss the case $\underline{\Phi} = \Theta(\overline{\Phi})$ under both Cluster Grid and Cluster Random models.

IV. Capacity bounds for the Cluster Grid Model

Using a combination of geometric and percolation arguments, we will show that:

Theorem 2: Under the *cluster-sparse* regime, the per-node throughput in the Cluster Grid model is upper-bounded, in order sense, by

$$\lambda = O\left(\frac{L\sqrt{\mu}}{n}\right) \qquad \text{if} \quad d_c\sqrt{\mu} = \omega(\log n) \tag{2}$$

$$\lambda = O\left(\frac{\sqrt{m}}{n}\log n\right) \qquad \text{if} \quad d_c\sqrt{\mu} = O(\log n) \tag{3}$$

where $\mu = q\, s(d_c)$.

Proof: The proof of (2) is provided in Section IV-A. The proof of (3) is provided in Section IV-B.

A. *The cluster sparse regime with* $d_c\sqrt{\mu} = \omega(\log n)$

We consider a rectangle of size $d_c/2 \times L$ located in between two adjacent columns of clusters centres, as illustrated by the shaded area in Figure 2. We observe that the local intensity of nodes is $\Phi(\xi) = \Theta\left(q\, s(d_c)\right)$ at any point ξ within the considered rectangle. Actually, the maximum node density within the rectangle, denoted by Φ_P, is found at points located at distance $d_c/4$ from one cluster centre, such as point P in Figure 2.

We divide the above rectangle into squares Z of area $|Z|$ and edge length z. Note that, being the nodes distributed according to an IPP, the random variables representing the number of nodes falling in each square are independent (although they are not identically distributed).

We set $|Z|$ in such a way that the probability that an arbitrary square within the rectangle contains no node is larger than the critical probability $p_c^s \approx 0.59$ of independent site percolation in the square lattice [18]. Having chosen $p > p_c^s$, the above condition is satisfied when $e^{-\Phi_P|Z|} = p$, i.e., by setting $|Z| = -\frac{\log p}{\Phi_P}$. Notice that in this case the square edge $z = \Theta(1/\sqrt{\Phi_P}) = o(d_c/\log n)$. By this choice of $|Z|$, percolation results similar to those exploited in [2], and reported in Appendix A, guarantee w.h.p. the existence, within the considered rectangle, of at least one path (actually, $\Theta(d_c/z)$ non-overlapping paths) formed by empty squares and connecting the top edge with the bottom edge of the rectangle. An example of top-to-bottom crossing path is depicted in Figure 2. Such paths behave almost as straight lines, and in fact there exists (w.h.p.) at least one crossing path comprising $\Theta(L/z)$ squares of area $|Z|$ (see Appendix A).

We consider such a path, and observe that it divides the network area into two parts each comprising $m/2$ clusters. Our goal is to upper bound the amount of information $\mathcal{F}$ that can flow from left to right through the cut delimited by the considered path. Since there are w.h.p. $\Theta(n)$ end-to-end data flows necessarily routed across the cut, the per-node throughput can then be upper bounded by $\mathcal{F}/n$.

To evaluate $\mathcal{F}$ we first consider, for simplicity, the case in which the top-to-bottom crossing path is exactly a straight line, and later extend the analysis to an arbitrary crossing path comprising a number of empty squares $l_c = \Theta(L/z)$. As illustrated in Figure 3, we partition the left side of the network area into squares with geometrically increasing edge length. More specifically, we use squares of edge equal to z right in contact with the squares forming the empty path. To the left of them, we use squares of edge $2z$. To the left of the latter ones, we

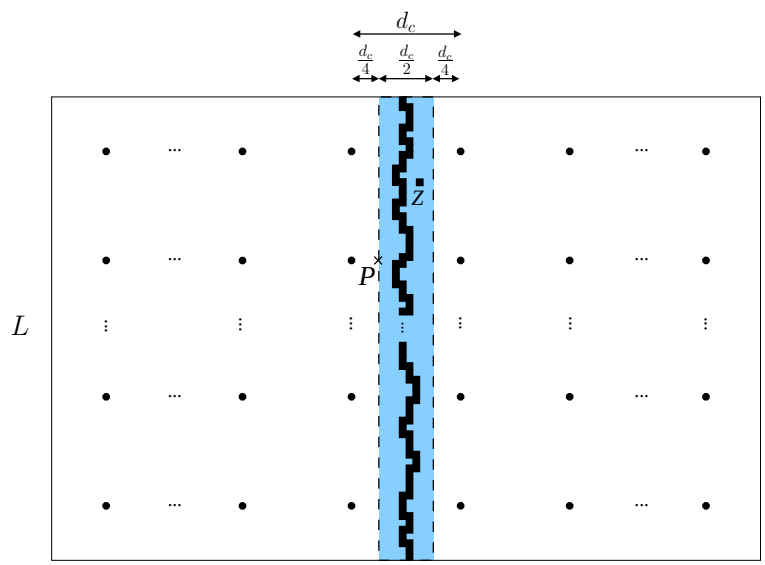


Fig. 2. Example of empty path cutting the grid in two halves

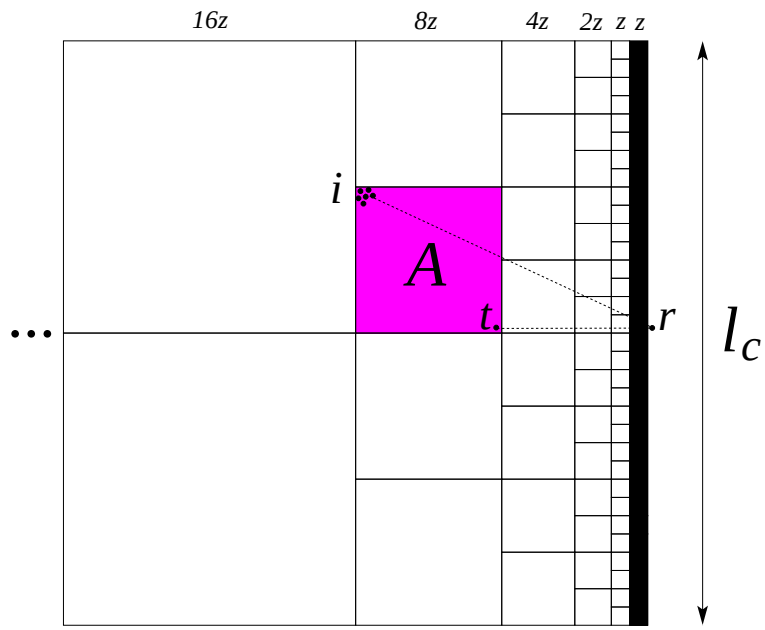


Fig. 3. Example of tessellation of the left portion of the network area with squares having geometrically increasing edge length

use squares of edge $4z$ and so on until we cover the entire left side of the network area.

Now, let us focus on a generic square A of edge $2^j z$, with $j \geq 0$, such as the shaded one in Figure 3, and let n_A be the number of nodes within it. We are going to show that the contribution

$\mathcal{F}_A$ of information sent by nodes in A through the cut is finite, even neglecting the interference produced by nodes residing in all of the other squares. This will allow us to upper bound $\mathcal{F}$ by the total number of squares forming our tessellation of the left side of the network area.

The best case for a transmitter t located within A is represented in Figure 3. Transmitter t is located in one corner of A, and its intended receiver r is located at the minimum possible distance right after the empty path, which is $2^j z$. All other transmitters i (if any), acting as interferers, are located in the opposite corner, and their distance from node r is equal to $\sqrt{5}\ 2^j z$. It can be rather easily verified that it is not possible to improve the SINR at receiver j by moving the receiver to a different position within the right side of the network area (we omit the proof). We further assume that the above optimal configuration holds for all transmitters within A, although this is clearly not possible when $n_A > 1$. However this is not an issue here since we are interested in obtaining an upper bound to $\mathcal{F}$.

To evaluate $\mathcal{F}$, we first consider the simple case in which all nodes within A employ the same transmitting power P. Next we generalize the analysis to the case in which nodes employ different powers.

a) Equal transmitting powers: If there is only one transmitter in A, the contribution $\mathcal{F}_A$ of the square is trivially finite, being upper bounded by R_0. If there are multiple transmitters, in any number n_A (even an infinite number), we can upper bound $\mathcal{F}_A$ as:

$$\mathcal{F}_A \leq n_A \min\left(R_0, \log\left(1 + \frac{P(2^j z)^{-\gamma}}{N_o + (n_A - 1)P(\sqrt{5}\ 2^j z)^{-\gamma}}\right)\right) \leq$$
$$\leq n_A \log\left(1 + \frac{P(2^j z)^{-\gamma}}{N_o + (n_A - 1)P(\sqrt{5}\ 2^j z)^{-\gamma}}\right) \tag{4}$$

Using the fact that $\log(1+x) < x$, and neglecting (optimistically) the noise term we have

$$\mathcal{F}_A < \frac{n_A P(2^j z)^{-\gamma}}{(n_A - 1)P(\sqrt{5} 2^j z)^{-\gamma}} < 5^{\gamma/2}\ 2 = \Theta(1) \tag{5}$$

We emphasize that the above bound becomes loose when $(n_A - 1)P(\sqrt{5}\ 2^j z)^{-\gamma} = o(1)$, since in this case the ambient noise N_0 would be the dominant term in the denominator of (4). In such a case, we could obtain a tighter upper bound by neglecting the interference term, and applying again the inequality $\log(1+x) < x$, we would get:

$$\mathcal{F}_A < \frac{n_A P(2^j z)^{-\gamma}}{N_o} = \Theta(n_A P(2^j z)^{-\gamma}) \tag{6}$$

Hence we obtain the following upper bound:

$$\mathcal{F}_A = \Theta\left(\min(1, n_A P(2^j z)^{-\gamma}\right) \tag{7}$$

In this paper we are mainly interested in the performance of *interference limited* systems, in which the noise term can indeed be neglected in the expression of SINR, at least for transmitters located in proximity of the crossing path (i.e., in squares with $j = 0$). Therefore we will use the bound (5) instead of the more general (7), and leave to future work the analysis of *noise limited* systems (or combined *interference-* and *noise-limited* systems).

b) Different transmitting powers: Now we show that no gain can be achieved by employing different transmitting powers within A. Again, if there is only one transmitter, there is nothing to prove, since the rate is limited to R_0. In the case of multiple transmitters, let $\hat{P}$ be the maximum power employed by nodes within A (this value can be arbitrarily large). We define a set of power classes, indexed by i, such that a node is declared to belong to power class i ($i = 1, 2, \ldots$) if its transmitting power falls in the interval $(\hat{P}/2^i, \hat{P}/2^{i-1}]$. Let n_i be the number of transmitters in A belonging to power class i, and $\mathcal{F}_A^i$ be their contribution to $\mathcal{F}_A$.

Let w be the index of the class for which the quantity $n_w \hat{P}/2^w$ is maximum among all classes. We separately analyze the contribution $\mathcal{F}_A^w$ from the contribution due to all other classes. An upper bound to $\mathcal{F}_A^w$ can be obtained assuming that there is no interference produce by nodes belonging to classes other than w. Moreover, we consider the ideal case in which, in addition to the optimal nodes' configuration shown in Figure 3, for any transmitter in class w the useful signal is sent at the maximum power $\hat{P}/2^{w-1}$, whereas all other interfering nodes in class w transmits at the minimum power $\hat{P}/2^w$. We obtain

$$\mathcal{F}_A^w \leq n_w \min\left(R_0, \log\left(1 + \frac{\hat{P}/2^{w-1}(2^j z)^{-\gamma}}{N_o + (n_w - 1)\hat{P}/2^w(\sqrt{5}\ 2^j z)^{-\gamma}}\right)\right) \leq$$

$$\leq n_w \log\left(1 + \frac{\hat{P}/2^{w-1}(2^j z)^{-\gamma}}{N_o + (n_w - 1)\hat{P}/2^w(\sqrt{5}\ 2^j z)^{-\gamma}}\right)$$

Similarly to the derivation of (5), we obtain that the contribution $\mathcal{F}_A^w$ is finite:

$$\mathcal{F}_A^w < \frac{n_w \hat{P}/2^{w-1}(2^j z)^{-\gamma}}{(n_w - 1)\hat{P}/2^w(\sqrt{5}2^j z)^{-\gamma}} < 5^{\gamma/2}\ 4 = \Theta(1)$$

For any other class $i \neq w$, we instead optimistically neglect the interference due to transmitters within the same class. Moreover, we assume that nodes belonging to class i transmits at the maximum power $\hat{P}/2^{i-1}$, whereas nodes belonging to class $k \neq i$ transmits at the minimum power $\hat{P}/2^k$. By so doing we obtain the upper bound:

$$\sum_{i \neq w} \mathcal{F}_A^i < \sum_{i \neq w} n_i \log\left(1 + \frac{\hat{P}/2^{i-1}(2^j z)^{-\gamma}}{N_o + \sum_{k \neq i} n_k \hat{P}/2^k(\sqrt{5}\ 2^j z)^{-\gamma}}\right)$$

Using again $\log(1+x) < x$, and ignoring the impact of N_0, we have

$$\sum_{i \neq w} \mathcal{F}_A^i < \sum_{i \neq w} \frac{n_i \hat{P}/2^{i-1}(2^j z)^{-\gamma}}{\sum_{k \neq i} n_k \hat{P}/2^k(\sqrt{5}\ 2^j z)^{-\gamma}}$$

Now we observe that, for any $i \neq w$,

$$\sum_{k \neq i} n_k \hat{P}/2^k(\sqrt{5}\ 2^j z)^{-\gamma} \geq \sum_{k \neq w} n_k \hat{P}/2^k(\sqrt{5}\ 2^j z)^{-\gamma}$$

owing to the definition of w. Hence we can write,

$$\sum_{i \neq w} \mathcal{F}_A^i < \frac{\sum_{i \neq w} n_i \hat{P}/2^{i-1}(2^j z)^{-\gamma}}{\sum_{k \neq w} n_k \hat{P}/2^k(\sqrt{5}\ 2^j z)^{-\gamma}} = 5^{\gamma/2}\ 2 = \Theta(1)$$

It follows that the overall rate produced by square A is finite:

$$\mathcal{F}_A = \mathcal{F}_A^w + \sum_{i \neq w} \mathcal{F}_A^i < 5^{\gamma/2}\ 6 = \Theta(1) \tag{8}$$

Notice that we have assumed, also in this case, that $\mathcal{F}_A$ is limited by the interference term. When $\sum_i n_i \hat{P}/2^i(2^j z)^{-\gamma} = o(1)$ a better bound is obtained by neglecting the interference term, obtaining:

$$\mathcal{F}_A < \frac{\sum_i n_i \hat{P}/2^{i-1}(2^j z)^{-\gamma}}{N_0} = \Theta\left(\sum_i n_i \hat{P}/2^{i-1}(2^j z)^{-\gamma}\right)$$

however we do not further investigate the *noise-limited* regime in our analysis.

From the above discussion, we conclude that, irrespective of the power strategy employed in the network, $\mathcal{F}$ can be upper bounded by the total number of squares in the considered tessellation of the area to the left of the empty path. Let l_c be the number of of empty squares forming the crossing path. Let $E_0 = l_c$ be the number of squares of edge z. The number of squares of edge $2z$ is $E_1 = E_0/2$. In general, the number of squares of edge $2^j z$ is $E_j = E_0/2^j$. We should mention that it is not typically possible to fit an integer number of squares of any size along the network extension[3] L. However, it can be easily shown (we omit the proof) that rounding effects do not alter, in order sense, the upper bound obtained in the case L can be divided perfectly by lengths $2^j z$.

Since numbers E_j form a geometric series, the total number of squares approaches $2E_0$ as the maximum index j tends to infinity (notice that the maximum index j grows as $\log n$). Considering

[3]In practice, one can build $\lfloor L/(2^j z) \rfloor$ squares plus at most one rectangle of vertical size smaller than $2^j z$, and repeat the same arguments.

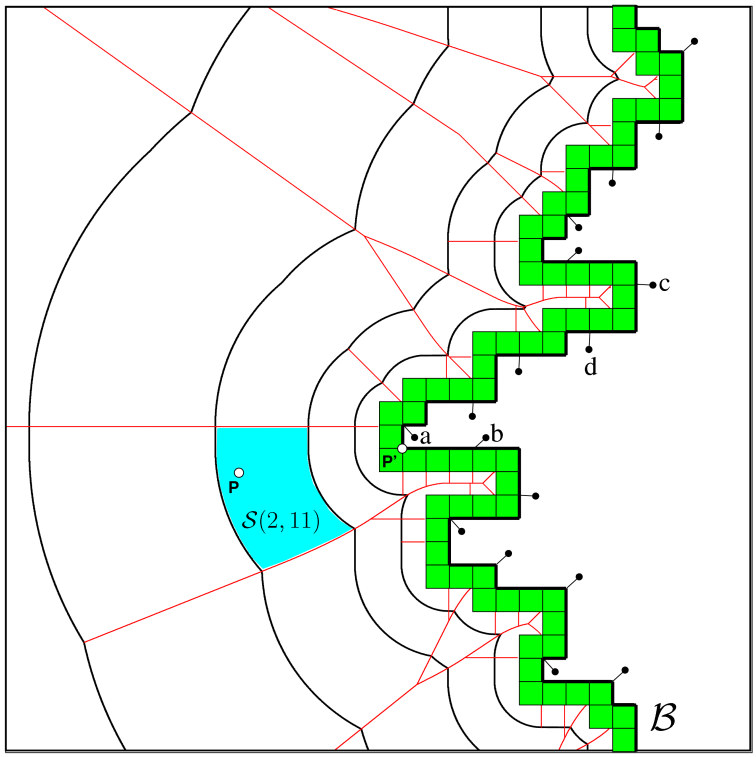


Fig. 4. Example of construction of sets $\mathcal{S}(j,i)$ in the network area to the left of a general top-to-bottom crossing path. In the case $j=2$, border $\mathcal{B}$ is partitioned into groups of adjacent edges with cardinality 4.

that $l_c = \Theta(L/z)$, $z = \Theta(1/\sqrt{\Phi_P})$, $\Phi_P = \Theta(q\, s(d_c))$, and that there are n flows traversing the cut, the final per-node throughput is upper-bounded, in order sense, by $\lambda = O(L\sqrt{q\, s(d_c)}/n)$.

■

We now extend the analysis to a generic path crossing the network area from top to bottom and comprising $l_c = \Theta(L/z)$ empty squares of edge z. Following the same rationale illustrated in Figure 3, we partition the network area to the left of the crossing path into sets $\mathcal{S}(j,i)$ each providing a contribution $O(1)$ to the information flow through the cut, and then evaluate the number of sets $\mathcal{S}(j,i)$.

To build sets $\mathcal{S}(j,i)$, we consider the border $\mathcal{B}$ separating the crossing path from the right part of the network area (this border is represented in Figure 4 by a thick solid line). For each point P in the left part of the area, we identify the point $P' \in \mathcal{B}$ at minimum euclidean distance from P, and denote such minimum distance by $d_{P,\mathcal{B}}$ (see example in Figure 4). Now, distance $d_{P,\mathcal{B}}$ belongs to one of intervals $(2^j z, 2^{j+1} z]$, with $j \geq 0$, and this provides us with the index j

used to classify point P into one of sets $\mathcal{S}(j,i)$.

Given the value of j, we partition border $\mathcal{B}$ into groups of adjacent edges each comprising 2^j edges of length z, and progressively number such groups using index i along border $\mathcal{B}$, from the top to the bottom edge. We then associate to point P the index i of the group comprising point P' [4]. For the example in Figure 4, point P is characterized by $j = 2$. Moreover, point P' belongs to the 11th group of cardinality 4 constructed along border $\mathcal{B}$ from the top to the bottom edge (comprising edges in between points a and b indicated in Figure 4) . The set of all points belonging to set $\mathcal{S}(2,11)$ (including P) is represented in Figure 4 by a shaded region.

We observe that, in the case of a straight crossing path, sets $\mathcal{S}(j,i)$ are exactly congruent to the squares of geometrically increasing edges illustrated in Figure 3. However, for a general crossing path the number of non-void sets $\mathcal{S}(j,i)$ can be smaller than the number of squares build in the case of a straight path. For example, in Figure 4 the set $\mathcal{S}(2,7)$ anchored to the group of edges between points c and d is empty. We conclude that the cardinality of sets $\mathcal{S}(j,i)$ is upper bounded by $2E_0$.

It remains to show that each set, considered in isolation, provides a contribution $O(1)$ to the information flow through the cut, irrespective of its shape and of the number of transmitters in it. This can be done following the same rationale adopted in the case of squares. Indeed, the minimum distance between a transmitter in set $\mathcal{S}(j,i)$ and a receiver r across the cut is, by construction, $2^j z$. Moreover, by triangular inequality the maximum distance between any interferer belonging to the same set and receiver r is at most $2^{j+1}z + 2^j z = 3 \cdot 2^j z$. Hence we can repeat the same derivation of equations (5) and (8) using 9 in place of 5. We conclude that (2) holds also for a general crossing path of length $l_c = \Theta(L/z)$, whose existence is guaranteed by the percolation arguments in Appendix A.

■

B. The cluster sparse regime with $d_c\sqrt{\mu} = O(\log n)$

This case occurs when the local intensity of nodes in the mid region between two adjacent cluster centres is so low that the edge length $z = \Theta(1/\sqrt{\Phi_P})$ becomes comparable or even larger

[4]Edge vertices belong to the group having the smallest index i. Moreover, if there multiple points P' at minimum distance form P, we take the P belonging to the group having the smallest index i

than d_c, so that (2) no longer applies. In this case we repeat the same construction as before, selecting a rectangle of size $d_c/2 \times L$ in between two adjacent columns of clusters centres (Figure 2), and divide it into squares Z of area $|Z|$ and edge length z. However in this case, irrespective of $\underline{\Phi}$, z is chosen to be $z = \frac{d_c}{\kappa \log n}$, where $\kappa > 0$ is a suitable constant.

Notice that, when $d_c\sqrt{\mu} = \Theta(\log n)$, the probability p that an arbitrary square within the rectangle contains no node can be made larger than the critical probability $p_c^s \approx 0.59$ by setting $|Z| > -\frac{\log p_c^s}{\Phi_P}$, i.e., by selecting κ large enough; when $d_c\sqrt{\mu} = o(\log n)$, instead, $p \to 1$ as n increases for any choice of $\kappa > 0$.

By an appropriate choice of κ, the same percolation results reported in Appendix A, guarantee (w.h.p.) also in this case the existence, within the considered rectangle, of at least one path (actually, $\Theta(\log n)$ non-overlapping paths) formed by empty squares and connecting the top edge with the bottom edge of the rectangle. Thus repeating the same arguments of Section IV we obtain that the per-node throughput is upper-bounded by $\lambda = O\left(\frac{L \log n}{n\, d_c}\right) = O\left(\frac{\sqrt{m}}{n} \log n\right)$. This concludes the proof of Theorem 2. ■

V. Capacity bounds for the Cluster Random Model

For the Cluster Random Model, we obtain upper bounds very close to those obtained for the Cluster Grid model.

Theorem 3: Under the *cluster-sparse* regime, the per-node throughput in the Cluster Random model is upper-bounded, in order sense, by

$$\lambda = O\left(\frac{L\sqrt{\mu}}{n}\right) \qquad \text{if} \quad d_c\sqrt{\mu} = \omega(\log n) \tag{9}$$

$$\lambda = O\left(\frac{\sqrt{m}}{n} \log n\right) \qquad \text{if} \quad d_c\sqrt{\mu} = O(\log n) \tag{10}$$

where $\mu = q\, s(d_c) \log n$.

Proof: The above upper bounds to the per-flow throughput are again obtained by evaluating the network flow through a cut traversed by $\Theta(n)$ end-to-end flows, which approximately divides the network area into two halves. Also in this case, the idea is to find an empty path crossing the network area from the top to the bottom edge, and lying in a region of minimal node density. However, here the approach is made slightly more complicated by the randomness of the clusters positions. We proceed in two steps: first we identify a large crossing path formed by big squares in which there are no cluster centres. Then we build a thinner crossing path, nested

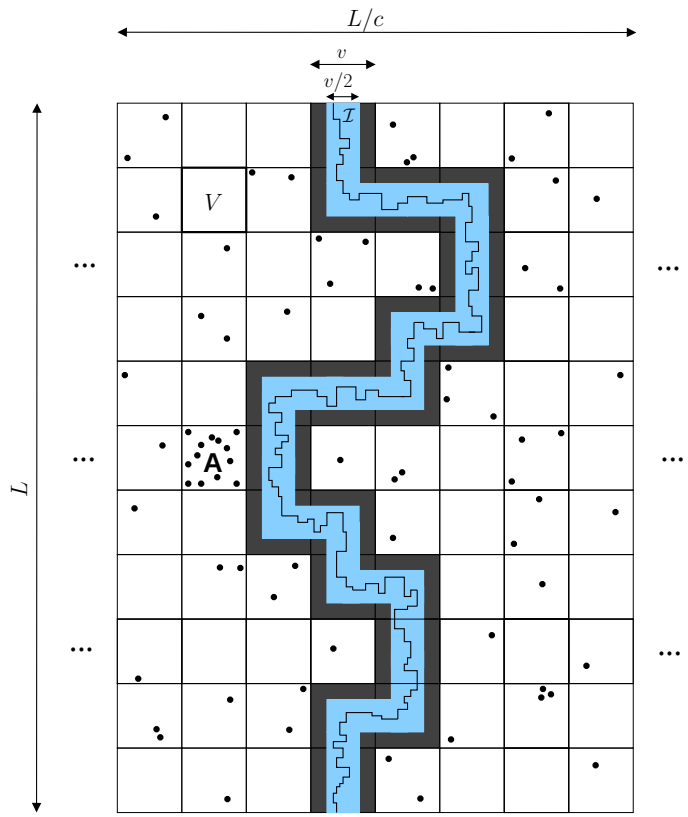


Fig. 5. Construction of a top-to-bottom crossing path nested in a larger top-to-bottom crossing path in which there are no cluster centres (represented by dots).

in the previous path, formed by smaller squares in which there are no nodes, and to which we can apply the same results derived in Section IV. Figure 5 offers a graphical representation of this construction. We now formalize the above idea and compute a simple upper bound to the resulting network capacity.

In the first step, we use squares V of area $|V|$ and edge length v. Similarly to what has been done in Section IV, we set $|V|$ in such a way that the probability that an arbitrary square contains no cluster centre is larger than the critical probability $p_c^s \approx 0.59$ of independent site percolation in the square lattice. Having chosen $p > p_c^s$, the above condition is satisfied when $e^{-\phi_c |V|} = p$, i.e., by setting $|V| = -\frac{\log p}{\phi_c}$. Notice that $v = \Theta(d_c)$.

Then the same percolation arguments used before allows to say that, in a rectangle $L/c \times L$ embedded in the network area (c is an arbitrary constant larger than 2), one can find w.h.p. at least one top-to-bottom crossing path $\mathcal{P}$ formed by $l_v = \Theta(L/v)$ squares V in which there are no

cluster centres. Notice that, for any $c > 2$, there are w.h.p. $\Theta(n)$ end-to-end data flows necessarily routed across the path. We remark that crossing path $\mathcal{P}$ depends on the cluster centres' positions $\mathbf{C}$.

Focusing on a given crossing path $\mathcal{P}$, we consider the inner, centered path $\mathcal{I}$ of width $v/2$ (see Figure 5). Since, by construction, no cluster centres fall within $\mathcal{P}$, the density of nodes at every point ξ of $\mathcal{I}$ can be upper bounded w.h.p. by a constant value Φ_P (specified later). Similarly to what has been done in Section IV, $\mathcal{I}$ is then partitioned into squares Z' of area $|Z'|$ and edge length z', and we look for a top-to-bottom crossing path within $\mathcal{I}$ made of empty squares Z'. Since the intensity at every point $\xi \in \mathcal{I}$ is dominated by Φ_P, the random variables denoting the number of nodes in different squares Z' are dominated by i.i.d variables distributed according to a Poisson distribution with mean $\Phi_P|Z'|$. Thus we can apply again the percolation results of Appendix A[5]. In particular, by an appropriate choice of z', the (conditional) probability that there is no top-to-bottom crossing path of squares Z' within $\mathcal{I}$ decreases exponentially to zero as n goes to infinity. Moreover, there exists at least one crossing path formed by a number $l'_c = \Theta(L/z')$ of empty squares of edge z' (using again the same argument as in Appendix A).

We conclude that, w.h.p., the above two nested crossing paths can be found. Then we can apply exactly the same techniques described in Section IV to upper bound the capacity through the inner crossing path (the one formed by squares Z'). The only difference with respect to the previous calculation lies in the size of z', which is directly related to the maximum node density Φ_P within path $\mathcal{I}$.

Differently from the Cluster Grid model, in which cluster centres are regularly spaced, we have to account for the possibility that some squares V right in contact to the outer crossing path $\mathcal{P}$ (for example, square $\mathbf{A}$ in Figure 5) are highly populated by cluster centres, with a consequent increase of the node density within $\mathcal{I}$. A simple upper bound to Φ_P can be obtained considering that, by corollary 1, uniformly over the entire tessellation of the network area, all squares V contains $O(\log m) = O(\log n)$ cluster centres. Under the optimistic assumption that all squares contain $\Theta(\log n)$ cluster centres, the maximum node density within $\mathcal{I}$ is $\Phi_P = \Theta(q\, s(d_c) \log n)$. Similarly to Section IV, we have to distinguish the two cases in which $d_c\sqrt{\Phi_P} = \omega(\log n)$ or

[5]A top-to-bottom crossing path does not exist only if at least one of the $\Theta(L/z')$ nodes on one side of path $\mathcal{I}$ is connected, in lattice $\mathcal{L}_{\boxtimes}$, to a node on the other side of $\mathcal{I}$. Hence we can repeat exactly the same arguments of Appendix A.

	cluster-dense (LB=UB)	*cluster-sparse* (LB) (from [7])	*cluster-sparse* (UB)
Cluster Grid	$\frac{1}{\sqrt{n}}$	$\max\left\{\frac{L\sqrt{qs(d_c)}}{n}, \frac{\sqrt{m}}{n}\right\}$	$\max\left\{\frac{L\sqrt{qs(d_c)}}{n}, \frac{\sqrt{m}}{n}\log n\right\}$
Cluster Random	$\frac{1}{\sqrt{n}}$	$\max\left\{\frac{L\sqrt{qs(d_c\sqrt{\log n})}}{n}, \frac{\sqrt{m}}{n}\right\}$	$\max\left\{\frac{L\sqrt{qs(d_c)\log n}}{n}, \frac{\sqrt{m}}{n}\log n\right\}$

TABLE II

THE PER-FLOW THROUGHPUT ACHIEVABLE IN DIFFERENT CASES. LB (UB) STANDS FOR LOWER BOUND (UPPER BOUND).

$d_c\sqrt{\Phi_P} = O(\log n)$, obtaining upper bounds to the per-node capacities given by (9) and (10), respectively. We observe that the above upper bounds for the Cluster Random model differs by a factor at most $\sqrt{\log n}$ from the corresponding upper bounds obtained for the Cluster Grid model.

■

VI. THE CLUSTER-DENSE REGIME

In this case, as proved in Section III, $\underline{\Phi} = \Theta(\overline{\Phi}) = \Theta(\frac{n}{L^2})$ under both Cluster Grid and Cluster Random models, hence $\Phi(\xi) = \Theta(\frac{n}{L^2})$ at any point $\xi \in \mathcal{O}$. In [7] we show that in this case it is possible to achieve the upper bound $\lambda = O(1/\sqrt{n})$ valid for arbitrary node placement (see [1] and extensions in [15]). We observe that an alternative proof of this bound can be obtained using the technique described in this paper, i.e., looking for a top-to-bottom crossing path in a rectangle $L/c \times L$ ($c > 2$) arbitrarily placed in the network area, and taking $\Phi_P = \overline{\Phi}$ as the maximum node density within it.

VII. SUMMARY OF RESULTS

Table VII summarizes the maximum achievable per-flow throughput (in order sense) under the *cluster-dense* ($\alpha < \nu/2$) and *cluster-sparse* ($\alpha > \nu/2$) regimes, for both Cluster Grid and Cluster Random models. The table reports the upper bounds (UB) obtained in this paper, together with the corresponding lower bounds (LB) derived in [7]. We observe that the lower bound coincides with the upper bound in the *cluster-dense* regime. In the *cluster-sparse* regime, upper and lower bounds differ at most by a poly-log factor, under the assumption that the system is limited by interference.

Lower bounds have been derived in [7] by introducing a class of scheduling and routing schemes which allows to achieve the capacity reported in Table VII. The main idea in [7] is to

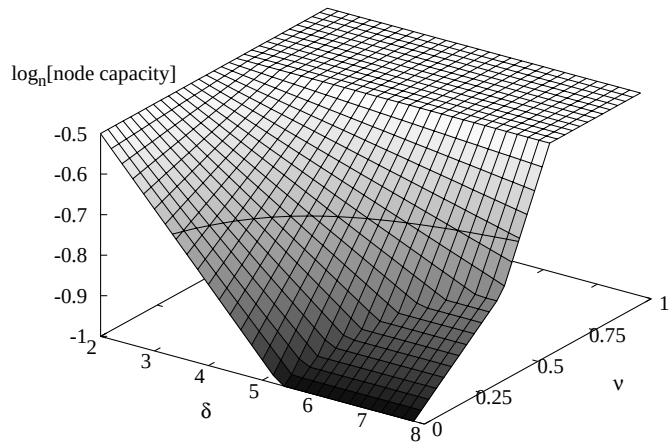


Fig. 6. Per-node throughput (in log_n scale) as a function of δ and ν, in the case $\alpha = 0.3$

extract a subset of nodes distributed according to an homogeneous Poisson process with intensity equal to the minimum node density $\underline{\Phi}$ over the area, and use this subset as the main transport infrastructure (highway system) of the network. The main challenge is to design clever scheduling and routing scheme to exploit the capacity provided by the main infrastructure avoiding the formation of bottlenecks while traffic moves towards the highway system from network regions having higher node density.

A. Graphical representation of results

To illustrate graphically our results, we consider the interesting case of functions $s(\rho)$ whose tail decays as a power-law: $s(\rho) \sim \rho^{-\delta}$, for $\delta > 2$. For the Cluster Grid model, we obtain that, in the cluster-sparse regime ($\alpha > \nu/2$), $\mu = \Theta(n^{1-\nu-\delta(\alpha-\nu/2)})$. Hence applying (2) the per-node throughput is $\lambda = O\left(n^{\alpha-(\nu+1)/2+(\alpha-\nu/2)\delta/2}\right)$, provided that $2(\nu-1)+(2\alpha-\nu)(\delta-2) < 0$. In this case the width z of the empty path is $\Theta(n^{\tau})$, where $\tau = (\nu-1)/2 + (\alpha-\nu/2)\delta/2$. We remark that, if $\tau > 0$, nodes get increasingly far apart in low-density regions, so our bound is tight only when nodes can compensate for the distance scaling up the emitted power (as $n^{\tau\gamma}$). Otherwise the network is *noise-limited* and the proposed bound becomes loose. For $\tau < 0$, instead, the network is *interference-limited* even if the nodes' transmission power is bounded.

In Figure 6 we have reported, using a $\log_n$ vertical scale, the per-node capacity for fixed

$\alpha = 0.3$, letting both δ and ν vary. Notice that on this scale we can neglect $\log n$ factors, thus results under the Cluster Random model look the same. We observe that the *cluster-dense* regime, in which $\lambda = O(1/\sqrt{n})$, occurs for all $\nu > 0.6$, independently of δ. In the *cluster-sparse* regime, the per-node capacity decreases for increasing values of δ and decreasing values of ν. When $2(\nu - 1) + (2\alpha - \nu)(\delta - 2) > 0$, the per-node capacity is $\lambda = O(n^{\nu/2-1} \log n)$ and approaches $1/n$ as ν tends to 0. At last, points below the contour curve drawn on the surface at $\lambda = n^{0.7}$ are characterized by $\tau > 0$, hence in this region the network is *noise-limited* if the transmission power is bounded.

VIII. Conclusions

In this paper we have studied the asymptotic capacity of clustered, random networks in which the local intensity of the node process can vary significantly across the network area, determining orders of magnitude difference between high-density and low-density regions. Using a combination of geometric and percolation arguments, we have obtained upper bounds to the per-node capacity which are tight for interference limited systems. Additional work is needed to obtain tight upper bounds also for noise-limited systems.

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APPENDIX A
PERCOLATION THEORY RESULTS

In this Section, we provide the percolation theory arguments that are needed to show the existence of an empty path of length $\Theta(L)$ and width $\Omega(z)$ cutting the network area in two regions of area $\Theta(L^2)$.

Let $\mathcal{L}_\square = \mathbb{Z}^2$ be the planar square lattice, and let $\mathcal{L}_\boxtimes$ be the graph with vertex set $\mathbb{Z}^2$ in which we add both diagonals to each face of $\mathcal{L}_\square$, so that any two vertices at Euclidean distance 1 or $\sqrt{2}$ are adjacent. Each vertex of $\mathbb{Z}^2$, also called a site, is in one of two states: open or closed. We consider the product probability measure $\mathbb{P}_p$ in which each site is open independently with probability p (equivalently, closed with probability $1-p$).

For $\mathcal{L} = \mathcal{L}_\square$ or $\mathcal{L}_\boxtimes$, the open cluster C_u containing the generic vertex u is the set of open vertices that may be reached from u by a path in the graph $\mathcal{L}$ all of whose vertices are open. The number of vertices in C_u is denoted by $|C_u|$. The radius $\mathrm{rad}(C_u)$ of the open cluster C_u is the maximum distance (in graph-theoretic sense) between u and other vertices belonging to C_u.

Let $p_c^s(\mathcal{L}_\square)$ and $p_c^s(\mathcal{L}_\boxtimes)$ be the critical probabilities for site percolation on $\mathcal{L}_\square$ and $\mathcal{L}_\boxtimes$, respectively. We recall that the critical probability is defined as

$$p_c^s = \sup\{p : \mathbb{P}_p(|C_u| = \infty) = 0\}$$

It was proved by Russo [16] that

$$p_c^s(\mathcal{L}_\square) + p_c^s(\mathcal{L}_\boxtimes) = 1 \tag{11}$$

Below the critical probability, a special case of the general result of Menshikov [17] implies that for $\mathcal{L}_\square$ or $\mathcal{L}_\boxtimes$ there is exponential decay of the cluster radius. This means that, for any $p < p_c^s$,

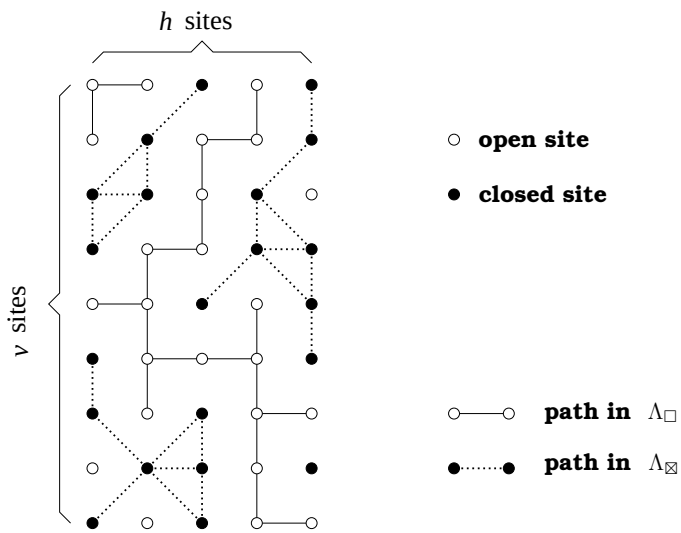

Fig. 7. Example of vertical open path in a rectangle with $h = 5$, $v = 9$. There cannot exist an horizontal path of closed sites in the same rectangle.

there exist $\sigma(p) > 0$ such that

$$\mathbb{P}_p(\mathrm{rad}(C_u) \geq m) < e^{-m\sigma(p)} \quad \text{for all } m \tag{12}$$

Let $R = [h] \times [v]$ be an h by v rectangle embedded in $\mathbb{Z}^2$. Let $V^o_{\mathcal{L}_\square}$ be the event that there is a path of open sites crossing R vertically in graph $\mathcal{L}_\square$, and $H^c_{\mathcal{L}_\boxtimes}$ be the event that there is a path of closed sites crossing R horizontally in graph $\mathcal{L}_\boxtimes$.

It can be easily seen (Figure 7) that, whatever the states of the sites in R, there is either a path of open sites crossing R vertically in $\mathcal{L}_\square$, or a path of closed sites crossing R horizontally in $\mathcal{L}_\boxtimes$. This implies that the probability that there does not exist a vertical crossing of open sites equals the probability that there exists an horizontal crossing of closed sites. In particular, in the case of independent state assignment to sites, we can write

$$1 - \mathbb{P}_p(V^o_{\mathcal{L}_\square}) = \mathbb{P}_p(H^c_{\mathcal{L}_\boxtimes})$$

Let $y_1 \dots y_v$ be the vertices along one vertical side of R. We denote by $\mathcal{H}_i$ the event that site y_i is connected by a path of closed sites to another vertex belonging to the opposite side of R. We have

$$\mathbb{P}_p(H^c_{\mathcal{L}_\boxtimes}) = \mathbb{P}_p(\cup_i\{\mathcal{H}_i\}) \leq \sum_{i=1}^{v} \mathbb{P}_p(\mathcal{H}_i)$$

A necessary but not sufficient condition for the event $\mathcal{H}_i$ to occur, is that the closed cluster [6] C_{y_i} has radius larger than or equal to h.

If we take a site percolation process in graph $\mathcal{L}_\square$ with $p > p_c^s(\mathcal{L}_\square)$, (11) implies that $1-p < p_c^s(\mathcal{L}_\boxtimes)$. This means that if we consider the site percolation process on graph $\mathcal{L}_\boxtimes$ in which each site is declared open independently with probability $p' = 1-p$, we are in the subcritical regime, and we have exponential decay of the radius of any cluster of open sites (closed sites in the original graph $\mathcal{L}_\square$) belonging to $\mathcal{L}_\boxtimes$. Hence using (12) we can write for all i,

$$\mathbb{P}_p(\mathcal{H}_i) \leq \mathbb{P}_{p'}(\mathrm{rad}(C_{y_i}) \geq h) < e^{-h\sigma(p')} \qquad \forall i$$

Putting things together, we obtain the inequality

$$1 - \mathbb{P}_p(V^o_{\mathcal{L}_\square}) \leq v e^{-h\sigma(1-p)} \tag{13}$$

Armed with the above result, we go back to the problem of finding an empty path of width $\Omega(z)$ crossing the network area from the top to the bottom edge. Each square Z within the rectangle $d_c/2 \times L$ can be mapped on a vertex of the square lattice, belonging to a rectangle of vertices having $h = d_c/(2z)$ and $v = L/z$. The probability p that a vertex is open equals the probability that the corresponding square Z does not contain any node. We can thus apply directly (13) to obtain the probability that there is no top to bottom crossing path formed by empty squares. Such probability tends to zero exponentially with n provided that $h\sigma > \log v$. We remark that (as required in the derivation of (3) and (10)) the latter inequality can be satisfied also when $h = \kappa \log n$, by choosing κ large enough (notice that $\log v$ is proportional to $\log n$ as well).

If, for the chosen value of $p > p_c^s$, there is exponential decay of the probability that there is no top-to-bottom crossing path, one can show an even stronger result, namely that there is exponential decay of the probability that there are $N_c = o(h)$ vertex-disjoint crossing paths. This means that one can actually find a number N_c of distinct crossing paths (not sharing any vertex) proportional to the width h of the rectangle. To show this fact, we consider a value p^* such that $p_c^s < p^* < p$ (for example $p^* = (p + p_c^s)/2$. Since $p^* > p_c^s$, we have exponential decay of the probability that there is no top-to-bottom crossing path in the rectangle in which the probability

[6] similarly to the open cluster, the closed cluster C_u containing vertex u is the set of closed vertices that may be reached from u by a path of closed vertices

that a vertex is open equals p^*. For $p > p^*$, the event that there is a vertical crossing path is expected to be more likely to occur. Indeed, one can characterize the probability that the same event still occurs even if we alter the state of r arbitrary vertices. This happens only when there are at least $r+1$ vertex-disjoint crossing paths. By the site percolation version of Theorem 2.45 of [18], the probability $\mathbb{P}_p(N_c \leq r)$ that there are less than $r+1$ vertex-disjoint crossing paths can be upper bounded as

$$\mathbb{P}_p(N_c \leq r) \leq \left(\frac{p}{p-p^*}\right)^r v e^{-h\sigma(1-p^*)}$$

Denoting by $r = \beta\, h$ ($0 < \beta < 1$) the number of vertex-disjoint crossing paths, the above probability decays exponentially with h provided that $\frac{\beta p}{p-p^*} - \sigma(1-p^*) < 0$, which is satisfied for β small enough. Hence we have $N_c = \Theta(h)$. At last, since there are $h\, v$ total vertices in the rectangle, if we have more than $\beta\, h$ vertex-disjoint paths, at least one of them is formed by a number of vertices less than or equal to $v/\beta = \Theta(L/z)$.